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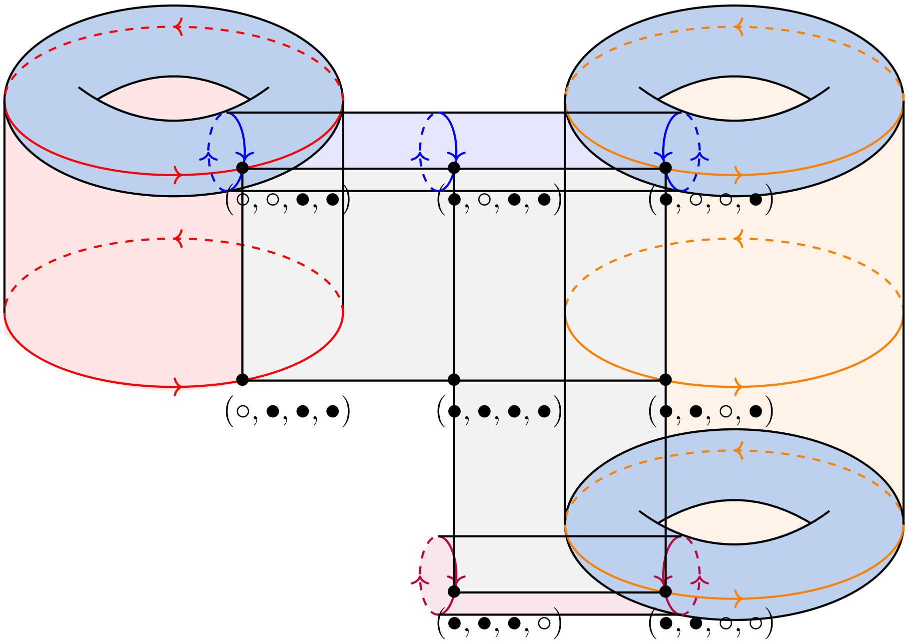
The geometry and topology of Artin groups



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Preface

Throughout the years I came in contact with many different mathematical subjects. In all of them, I enjoyed the reasoning's behind proofs, theorems, lemmas... The topics that surprised me the most were the theorems and cleverness in proofs in the abstract math. So that narrowed down the options for my master's thesis. I have chosen algebra, because I find it very interesting to wonder about groups, rings, etc. Even though the reader may already somewhat know what's in this project, I hope this person is still surprised or intrigued when reading my thesis. I enjoyed creating this, and I can genuinely say that it has been the most fun experiences I have ever had. Even though I did not create imposing new theorems, the smaller theorem, examples and proofs that I found myself gave me a bit of a "mathematical high".

Unfortunately at everything great comes an end; including working on this thesis and this preface, to end I want to thank people. First and also most I want to thank Tom De Medts, of course for assisting me throughout this year giving me sources to explore and helping me in finding proofs, examples, and insight. Also for making me curious about these subjects. Inspiring me in mathematics throughout the years by lecturing Algebra 1, Algebra 2 and linear Algebraic Groups, by not only explaining these topics but also by giving extra facts, interesting information and hence, making me eager to learn more. I thank all the professors and assistants who gave other mathematical courses throughout the years. Also, most importantly the staff of UGent especially the staff in the S5 and library, since that's where my master's thesis was made. I want to thank free tools like the program TeXstudio in which I made this. Also, the free application "languagetool" and the UGent to facilitate accommodation for my not to be underestimated dyslexia. Finally, I want to thank my parents, sisters, brother, my fellow students and friends from the S5 to encourage me during the past 5 year and to listen patiently to my exciting mathematical stories.

I am also grateful for all the luck and coincidence I had in my life to come to this point.

Contents

Introduction

In this master's thesis, Artin groups are studied, with an emphasis on right-angled Artin groups. Central to this work is the interplay between the algebraic structures and their geometric realizations.

Most of the work done in this thesis is a summary of papers/books of other people. The books/papers that were of the most use were primarily the book from Michael Davis [24], not only for its information about Coxeter complexes but also for its easy introductions into simplicial complexes. The papers by Huang [33, 32, 31] forms a basis of Chapter 5 and Chapter 6.

We start with an introduction to Coxeter groups and their Tits representations in Chapter 1, this Tits representation serves as the basis for constructing the *Davis complex* in Section 2.7. Using this complex, the *Salvetti complex* is constructed in Section 2.9, a topological space whose fundamental group is precisely the Artin group associated with the Coxeter group we start with. A different equivalent definition (Definition 2.9.2) for the Salvetti complex can be made via a graph product for right-angled Artin groups, we will use the same idea later to define the exploded Salvetti complex (Section 5.2).

An analogous concept to the Davis complex for a Coxeter group is the *Deligne complex* for an Artin group. In the case of a right-angled Artin group, this complex is equivalent to a (Tits-)building (Section 4.5), where the chambers correspond to the elements of the Artin group. We also discuss the unresolved problem of the $K(\pi, 1)$-conjecture, and what equivalent statements there are in terms of the Salvetti complex and the Deligne complex.

Furthermore, for right-angled Artin groups, we examine structures such as the *exploded Salvetti complex* and the *extension complex*. The extension complex brings us to the last chapter since it is of interest in the study of the quasi-isometric classifications of right-angled Artin groups. The universal cover of the exploded Salvetti complex, like that of the Salvetti complex, is a CAT(0)-

cube complex and has a natural connection to the geometric realization of the associated building. We will use this connection to prove some quasi-isometric properties of right-angled Artin groups. In this last chapter we will determine some classes of rigid right-angled Artin groups (i.e. they are quasi-isometric if and only if they are isomorphic). We will see that all right-angled Artin group with defining graph a tree of diameter at least 3 are quasi-isometric. Finely in Section 6.9 we will define a doubling argument on the defining graph of a right-angled Artin group such that the Artin group of our new graph is quasi-isometric to our Artin group we started with (Theorem 6.9.4). Some of the theorems we will discuss that are some what novel are the following.

- Theorem 2.12.1, where we prove some isomorphisms between Artin groups and quotients of fundamental groups. However, that fact that this theorem will be true came from [18, footnote page 149].
- Theorem 4.6.4, herein we prove that the Deligne complex of non-right-angled Artin groups are never buildings.
- Theorem 6.9.4 (ii) & (iii), in (ii) we will construct a quasi-isometry between the universal cover of the Salvetti complex of a right-angled Artin group A_Γ and that of another right-angled Artin group with defining graph a k-double of Γ. More importantly in (iii) we give a partial solution to the question described in Remark 6.3.6, that is, we prove that the extension complexes of these Artin groups are isomorphic.
- In Theorem 6.9.8 and Corollary 6.9.10, we prove that the k-doubling argument (Theorem 6.9.4) is insufficient to find quasi-isometries between any two right-angled Artin groups whose defining graphs are trees of diameter at least 3.

To become familiar with the various structures and topics addressed in this thesis, we suggest that the reader to initially study the complexes presented in Chapter 2. The discussion in Chapter 4 then shows the connections with buildings. Although Chapter 5 is technically dense, and many of its detailed properties are not required for the later sections, the reader may opt to proceed directly to Chapter 6. In this final chapter, Theorem 6.9.4 and the subsequent examples shows an application of the Salvetti complex.

1 Preliminaries

Coxeter groups and Artin groups are both defined by a graph with labeled edges, these graphs are sometimes called unoriented Dynken diagrams or Coxeter diagrams. In this Chapter we will introduce Artin groups and Coxeter groups by defining them in this way. We will also define a representation of a Coxeter group as a subgroup of $\mathsf{GL}_n(\mathbb{R})$.

1.1 Defining graph

Definition 1.1.1. Consider a finite simple graph Γ with vertex set $V(\Gamma) = \{s_i \mid i \in I\}$ for a certain index set I, where each edge has a label in $\mathbb{N}_{\geq 2}$. For two vertices $s_i, s_j \in V(\Gamma)$, if there is an edge between the two, we will denote $m_{ij} \in \mathbb{N}_{\geq 2} \sqcup \{\infty\}$ to be the label of this edge. If s_i and s_j are not connected, then we set $m_{ij} := \infty$. For $i = j$, we set $m_{ii} := 1$. By definition, we have $m_{ij} = m_{ji}$.

Remark 1.1.2. In other literature one regularly uses the convention to not draw edges of label 2 and draw edges between vertices where $m_{ij} = \infty$. That said in this thesis, two vertices will be non-adjacent if and only if $m_{ij} = \infty$.

Using a labeled graph we can give an explicit representation of these groups. In section 1.2 we will represent the Coxeter group differently as reflections of $\mathbb{R}^n$.

Definition 1.1.3. For Γ a graph as in Definition 1.1.1, we denote A_Γ for the associated *Artin group*[1], and W_Γ for the associated *Coxeter group*, defined as

[1] Artin groups are named after Emil Artin for his early work on braid groups. Jacques Tits developed this theory more generally, hence, they are also called "Artin-Tits groups".

follows

$$A_\Gamma := \left\langle s_i \in V(\Gamma) \mid \underbrace{s_i s_j s_i \cdots s_i s_j \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots s_j s_i \cdots}_{m_{ij} \text{ terms}} \right\rangle;$$

$$\begin{aligned} W_\Gamma &:= \left\langle s_i \in V(\Gamma) \mid \underbrace{s_i s_j s_i \cdots s_i s_j \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots s_j s_i \cdots}_{m_{ij} \text{ terms}},\ (s_i)^2 = \mathbb{1} \right\rangle \\ &= \left\langle s_i \mid (s_i s_j)^{m_{ij}} = \mathbb{1},\ (s_i)^2 = \mathbb{1} \right\rangle. \end{aligned}$$

We say that W_Γ and A_Γ have *type* or *defining graph* Γ and *rank* $n := |V(\Gamma)|$.

Example 1.1.4. (i) Consider $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$. We have that $W_\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $A_\Gamma = \mathbb{Z} \times \mathbb{Z}$. If $\Gamma := \overset{a}{\bullet} \quad \overset{b}{\bullet}$, then we change the products to free products, i.e. $W_\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $A_\Gamma = \mathbb{Z} * \mathbb{Z}$.

(ii) The symmetric group on 3 elements is a Coxeter group of type $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$. The Artin group of type Γ is the braid group on 3 strands. There has been done extensive research on braid groups. See also Example 3.2.2 and the paper about the braid group on four strands [17] in context of Artin groups.

Remark 1.1.5. (i) As in Example 1.1.4, we see that the Coxeter group is sometimes finite. All finite Coxeter groups have been classified in [22]. In contrast, every Artin group is infinite.

(ii) Whenever we say Λ a subgraph of Γ we always mean an induced subgraph, i.e. the largest subgraph in Γ that contains only the vertices $V(\Lambda)$.

(iii) Clearly, for every subgraph $\Lambda \subseteq \Gamma$ there exists a subgroup of the form $\langle s \mid s \in V(\Lambda)\rangle_{W_\Gamma} \leq W_\Gamma$. It happens to be the case that $W_\Lambda \cong \langle s \mid s \in V(\Lambda)\rangle_{W_\Gamma}$ ([21, Lemma 1.1.1]). Similarly, for Artin groups is $A_\Lambda \cong \langle s \mid s \in V(\Lambda)\rangle_{A_\Gamma} \leq A_\Gamma$ ([21, Corollary 3.25] and was first proven by van der Lek [50, Theorem 4.12]).

Definition 1.1.6. Consider an Artin group A_Γ and the associated Coxeter group W_Γ with rank n and defining graph Γ with edge labels $(m_{ij})_{ij}$.

(i) The Artin group A_Γ is of *finite type* (also called *spherical*) if W_Γ is finite and of *infinite type* if W_Γ is infinite.

(ii) The subgroups of the form W_Λ (respectively A_Λ) of W_Γ (respectively A_Γ) for a subgraph Λ of Γ are called *special subgroups*. If, in addition, W_Λ is finite we say that W_Λ (respectively A_Λ) is a *spherical subgroup*.

(iii) If for every pair $i, j \in I$ we have $m_{ij} \in \{2, \infty\}$, then we call W_Γ a *right-angled Coxeter group* and A_Γ a *right-angled Artin group* (in short RACG and RAAG, respectively). We will also call the defining graph Γ *right-angled*.

(iv) The *Coxeter matrix* M is the $n \times n$ matrix $M := (m_{ij})_{ij}$.

(v) A *special coset* of W_Γ (respectively A_Γ) is a coset of a special subgroup $W_\Lambda \leq W_\Gamma$ (respectively $A_\Lambda \leq A_\Gamma$). A *spherical coset* of W_Γ (respectively A_Γ) is a coset of a spherical subgroup $W_\Lambda \leq W_\Gamma$(respectively $A_\Lambda \leq A_\Gamma$). The set of special cosets of W_Γ (respectively A_Γ) is denoted as $W_\Gamma\mathcal{S}$ (respectively $A_\Gamma\mathcal{S}$) and the set of spherical cosets as $W_\Gamma\mathcal{S}^f$ (respectively $A_\Gamma\mathcal{S}^f$). Similarly, let $\mathcal{S}^f \subseteq \mathcal{P}(V(\Gamma))$ be the poset of spherical subsets of $V(\Gamma)$ (i.e. the subsets $\Lambda$of Γ for which W_Λ is finite). Hence, we have

$$W_\Gamma\mathcal{S}^f := \{gW_\Lambda \mid g \in W_\Gamma, \Lambda \text{ subgraph of } \Gamma \text{ and } W_\Lambda \text{ is finite}\};$$
$$A_\Gamma\mathcal{S}^f := \{gA_\Lambda \mid g \in A_\Gamma, \Lambda \text{ subgraph of } \Gamma \text{ and } W_\Lambda \text{ is finite}\}.$$

(vi) The Coxeter group and Artin group are of *FC-type*, if for every clique $\Lambda \subseteq \Gamma$ the group W_Λ is finite.

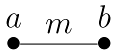
Throughout this thesis the following Lemma we will sometimes need.

Lemma 1.1.7. *Let W_Γ be a Coxeter group with defining graph Γ. Suppose $s_1 s_2 \ldots s_k = \mathbb{1}$ for $s_i \in V(\Gamma)$ with $\forall i, s_i \neq \mathbb{1}$, then k is even.*

Proof. Exercise. □

1.2 Tits representation

A Coxeter group W_Γ can be naturally represented as a group generated by a set of reflections in $\mathbb{R}^n$ with $n = |V(\Gamma)|$.

Example 1.2.1. Let $\Gamma :=$ 

. Consider $\tilde{a}$ and $\tilde{b}$ two straight lines in $\mathbb{R}^2$ such that they intersect in one point and have an angle of $\frac{\pi}{m}$ between them, see Figure 1.2.1. Consider the two reflection of $\mathbb{R}^2$ along these hyperplanes $\tilde{a}$ and $\tilde{b}$, these are bijective transformations of $\mathbb{R}^2$. The group generated by these two reflections is isomorphic to W_Γ.

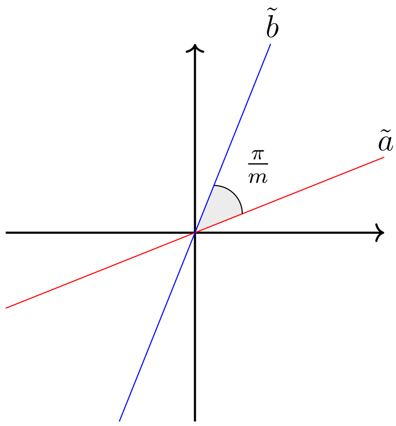


Figure 1.2.1: Reflection hyperplanes in $\mathbb{R}^2$.

We want to find a representation like in Example 1.2.1 for arbitrary Coxeter groups. A lot more information about reflection systems, pre Coxeter systems, Coxeter systems, affine reflection groups, etc. can be found in [24, Chaper 3].

Definition 1.2.2. For W_Γ a Coxeter group, the *Schläfli matrix* CM_Γ is the $n \times n$ ($n = |\Gamma|$) matrix $\mathrm{CM}_\Gamma := \left(-\cos\left(\frac{\pi}{m_{ij}}\right)\right)_{ij}$ where we set $\cos(\frac{\pi}{\infty}) := 1$.

Example 1.2.3. Let $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$ then the Schläfli matrix is

$$2CM_\Gamma = \begin{bmatrix} -2\cos(\pi) & -2\cos(\frac{\pi}{3}) \\ -2\cos(\frac{\pi}{3}) & -2\cos(\pi) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Definition 1.2.4 (Tits representation [39, Section 2])**.** Suppose W_Γ is a Coxeter group with rank n and $\{s_1, s_2, \ldots, s_n\} := V(\Gamma)$ the generating set of W_Γ and let CM_Γ be the Schläfli matrix. Consider $\mathbb{R}^n$ as a vector space with standard basis $\{e_1, e_2, \ldots, e_n\}$, then we define a linear map r_i for every s_i:

$$\begin{aligned} r_i : \mathbb{R}^n &\to \mathbb{R}^n : \\ & v \mapsto v - 2\left(v^T \mathrm{CM}_\Gamma e_i\right) e_i. \end{aligned}$$

One can verify (See [10, Proposition 4.1.2]) that every r_i is a bijective involutive linear map, that there is a hyperplane that is point-wise fixed by r_i and that the order of $r_i \circ r_j$ is m_{ij} (See [24, Lemma 6.12.3]), where m_{ij} is the label of the edge between vertices $s_i, s_j \in V(\Gamma)$.

Example 1.2.5. We carry on from Example 1.2.3. The maps r_1 and r_2 are the following.

$$\begin{aligned} r_1 : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} y - x \\ y \end{pmatrix}; \\ r_2 : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x \\ x - y \end{pmatrix}. \end{aligned}$$

The fix hyperplanes are $y = 2x$ and $y = \frac{1}{2}x$ respectively. However, in this case these maps are not orthogonal reflections (in the meaning that $v \mapsto -v$ if v was orthogonal to the fix hyperplanes). For this to be an orthogonal reflection we choose vectors α_1 and α_2 such that

$$\begin{aligned} \alpha_1 \cdot \alpha_2 &= (CM_\Gamma)_{12}; \\ \alpha_i \cdot \alpha_i &= (CM_\Gamma)_{ii} = 1. \end{aligned}$$

This is satisfied if we choose $\alpha_1 := (0, 1)^T$ and $\alpha_2 := (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$. Then we can define reflections as follows

$$\tilde{r}_i : \mathbb{R}^2 \to \mathbb{R}^2 : v \mapsto v - 2(\alpha_i \cdot v)\alpha_i.$$

These orthogonal reflections have reflection hyperplanes $y = 0$ and $y = \sqrt{3}x$. These two lines have an angle of $\pi/3$ between them.

Definition 1.2.6 (Orthogonal reflections)**.** The Tits representation gives us linear reflections (i.e. linear involutive maps that fix a hyperplane). If W_Γ is finite then one can prove ([21, page 598] and [18, page 143]), that we can find α_i (as in Example 1.2.5), such that W_Γ can be represented as orthogonal reflections. For finite Coxeter group we will always refer to these orthogonal reflections when we are talking about the Tits-representation.

When we will need these representation as reflections we will almost always be working with finite Coxeter groups (except for Section 3.3).

For more information about these reflection systems we recommend [24] and [1].

Theorem 1.2.7 ([39, Theorem 2.4.], [10, Theorem 4.1.3])**.** *For every Coxeter group W_Γ, its Tits representation is faithful (i.e. $W_\Gamma \cong \langle r_i \mid i \in \{1, 2, \dots, n\} \rangle_{\mathsf{GL}_n(\mathbb{R})} \leq \mathsf{GL}_n(\mathbb{R})$). Moreover, this isomorphism exactly sends $s_i \mapsto r_i$ for all $s_i \in V(\Gamma)$.*

Proof. See [10, Theorem 4.1.3]. □

Remark 1.2.8. Consider the Tits representation in Definition 1.2.4 of a Coxeter group W_Γ. Let v_s be the unique (up to sign) unit vector orthogonal to the fixed hyperplane of r_s. It could be that $\{v_s \in \mathbb{R}^{|V(\Gamma)|} \mid s \in V(\Gamma)\}$ is a linearly independent set. In this case we replace each matrix M_{r_s} (which is the matrix corresponding to the linear map $r_s : \mathbb{R}^n \to \mathbb{R}^n$) with its inverse transpose $M_{r_s}^{-T}$ (see [39, Remark 2.8]). From now on we assume without loss of generality that $\{v_s \in \mathbb{R}^n \mid s \in V(\Gamma)\}$ is linearly independent.

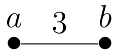
We will now define the fundamental cone, we will need this later to define the fundamental domain of this action (Definition 2.7.2 and Remark 2.7.4 (ii)). We will also need this in the discussion of the $K(\pi, 1)$ conjecture (Chapter 3).

Definition 1.2.9 (Simplicial cone)**.** Let $v_1, v_2, \dots, v_r$ a set of linear independent vectors in $\mathbb{R}^n$. The *simplicial cone* C generated by $v_1, v_2, \dots, v_r$ is the set of vectors that are linear combination of v_i with positive scalers, i.e.

$$C = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots \lambda_r v_r \mid \lambda_i \in \mathbb{R}^+\}.$$

Definition 1.2.10 (fundamental cone [39, Definition 2.6])**.** Suppose W_Γ is a Coxeter group with $n := |V(\Gamma)|$ and let $\{v_s \in \mathbb{R}^n \mid s \in V(\Gamma)\}$ be a set of vectors such that v_s is orthogonal to the fixed hyperplane in $\mathbb{R}^n$ of the linear map r_i (constructed in Definition 1.2.4). The *fundamental cone* of W_Γ is the following simplicial cone

$$\mathrm{cone}(\Gamma) := \left\{ \sum_{s \in V(\Gamma)} \lambda_s v_s \mid \lambda_s \in \mathbb{R}^+ \right\}.$$

Example 1.2.11. (i) Let $\Gamma :=$ 

, then we have two generators, they correspond to two reflection hyperplanes in $\mathbb{R}^2$ and are drawn in Figure 1.2.2. The space colored in red is the fundamental cone, the orbit if this space tiles the whole plane $\mathbb{R}^2$.

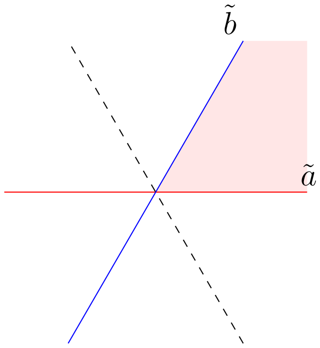
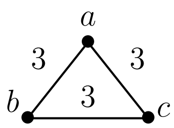


Figure 1.2.2: Reflection hyperplanes of W_Γ.

(ii) Now let $\Gamma :=$ 

, this is an infinite Coxeter group[2]. Despite that, there is still a way such that W_Γ can be represented as a group acting on $\mathbb{R}^2$ by reflections. We can choose three lines such that they all intersect each other in an $\frac{\pi}{3}$ angle, as seen in Figure 1.2.3. Here again the orbit of the space enclosed by all the three lines will tile the whole $\mathbb{R}^2$ plane. If we draw a vertex inside this enclosed space, and draw the orbit of this vertex we will get the Cayleygraph of W_Γ (more of this see section 2.7).

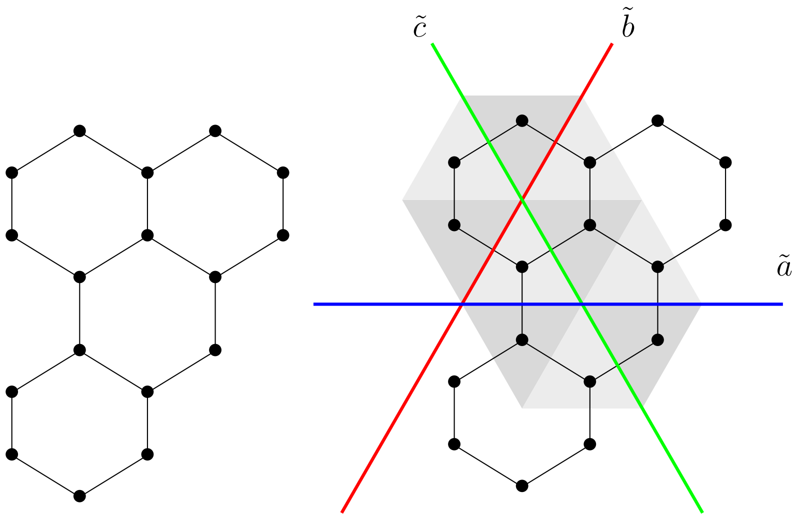


Figure 1.2.3: Reflection hyperplanes in $\mathbb{R}^2$.

[2]This Coxeter group is the "smallest" example of a non FC-type Coxeter group.

2

Artin groups and associated complexes

2.1 Simplicial complexes

Readers familiar with simplicial complexes, cube complexes, and geometric realizations may proceed directly to Section 2.5. We need the following definitions to define the Davis complex, Deligne complex, etc. which will be complexes showing geometric and topological properties of our Coxeter and Artin groups.

Definition 2.1.1. An *abstract simplicial complex* consists of a set S and a set of finite subsets $\mathcal{S} \subseteq \mathcal{P}(S)$, whose elements are called *simplices*. This set satisfies the following conditions:

(i) The intersection of two simplices is a simplex, i.e. $(\forall A, B \in \mathcal{S})(A \cap B \in \mathcal{S})$;

(ii) Every element s in the set S forms a simplex $\{s\}$, i.e. $(\forall s \in S)(\{s\} \in \mathcal{S})$;

(iii) Every subset of a simplex is a simplex, i.e. $(\forall T \in \mathcal{S})(\forall T' \subseteq T)(T' \in \mathcal{S})$.

A *k-simplex* is a subset $T \in \mathcal{S}$ with $|T| = k + 1$. The *dimension* of a simplex T is $|T| - 1$. A *subcomplex* of a simplicial complex is a subset $T \subseteq \mathcal{S}$ that also satisfies the axioms of an abstract simplicial complex (possibly for a smaller set $S' \subseteq S$). The *k-skeleton* $\mathcal{S}^{(k)}$ is the subcomplex of $\mathcal{S}$ consisting of all simplices of dimension $\leq k$. A *vertex* of $\mathcal{S}$ is a 0-simplex, and an *edge* is a 1-simplex.

Definition 2.1.2. Consider P a poset. *The abstract simplicial complex of a poset* P is the simplicial complex

$$\text{Flag}(P) := \{v \in \mathcal{P}(P) \mid \forall a, b \in v, a \leq b \vee a \geq b\}.$$

One can check that Flag(P) satisfies the axioms of Definition 2.1.1.

Example 2.1.3. Consider $\mathbb{R}^n$ with standard basis $e_1, e_2, \ldots, e_n$ and let $\mathbb{R}^n \supseteq \mathcal{S} := \{\{e_i\} \mid i \in \{1, 2, \ldots, n\}\} \cup \{\{e_i, e_j\} \mid i \neq j \in \{1, 2, \ldots, n\}\} \cup \cdots \cup \{\{e_1, e_2, \ldots, e_n\}\}$. We can visualize this as follows: a 0-simplex is a real point e_i in $\mathbb{R}^n$, a 1-simplex is a real line segment form e_i to e_j, etc. For $n = 3$, Figure 2.1.1 shows this visualization.

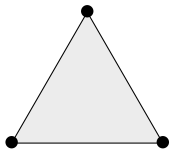


Figure 2.1.1: Geometric realization of $\mathcal{S}$ in $\mathbb{R}^3$.

Definition 2.1.4. The convex hull of the standard basis $e_1, e_2, \ldots e_n \in \mathbb{R}^n$ is called the *standard n-simplex* Δ_n. This is the abstract simplicial complex from Example 2.1.3. Consider $\mathcal{S}$ an abstract simplicial complex. The *geometric realization* $\mathrm{Geom}(\mathcal{S})$ of $\mathcal{S}$ consists of one copy of Δ_n for every n-simplex $T \in \mathcal{S}$. This copy, denoted as σ_T, is the span of the vertices $s_i \in T^{(0)}$, where the vertices of σ_T correspond to the singletons of T. If $T' \in \mathcal{S}$ and $T'^{(0)} \subseteq T^{(0)}$, then $\Delta_{|T'|}$ is attached to $\Delta_{|T|}$ in a natural manner, such that $\sigma_{T'}$ is a subcomplex of σ_T .

Definition 2.1.5. Let $\mathcal{S}$ be an abstract simplicial complex, then the *barycentric subdivision* of $\mathcal{S}$ is the abstract simplicial complex $\mathrm{Flag}(\mathcal{S})$, where $\mathcal{S}$ is seen as a poset for the inclusion, i.e.

$$\mathrm{Flag}(\mathcal{S}) := \{V \in \mathcal{P}(\mathcal{S}) \mid \forall A, B \in V, A \subseteq B \vee A \supseteq B\}.$$

Example 2.1.6. Consider the poset $P := \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. This poset also satisfies the axioms of a simplicial complex. The geometric realization $\mathrm{Geom}(\mathcal{S})$ of $\mathcal{S} := P$ is given in Figure 2.1.1. The geometric realization $\mathrm{Geom}(\mathrm{Flag}(\mathcal{S}))$ is the barycentric subdivision of $\mathrm{Geom}(\mathcal{S})$, given in Figure 2.1.2.

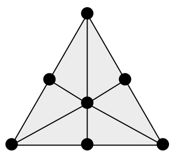


Figure 2.1.2: Barycentric subdivision of $\mathcal{S}$.

Definition 2.1.7. Suppose $\mathcal{S}$ is an abstract simplicial complex of a set S. If for every finite subset $T \subseteq S$ the condition: that if for every pair of elements $t_1, t_2 \in T$ that $\{t_1, t_2\} \in \mathcal{S}$, then we have that $T \in \mathcal{S}$, is satisfied, then we call $\mathcal{S}$ a *flag complex*, i.e.

$$(\forall T \subseteq S)\left(\left(\forall t_1, t_2 \in T, \{t_1, t_2\} \in S\right) \Rightarrow T \in \mathcal{S}\right).$$

Intuitively, a flag complex is an abstract simplicial complex $\mathcal{S}$ with the property that whenever we have a clique in $\mathcal{S}^{(1)}$, the complex includes the simplex spanned by those vertices. In other words, you fill this part of the geometric realization in with a simplex $\Delta_n \subseteq \mathbb{R}$.

Remark 2.1.8. Clearly, every abstract simplicial complex formed by a poset (see Definition 2.1.2) is a flag complex.

Definition 2.1.9 (CW-complex)**.** Consider $D^k := \{x \in \mathbb{R}^k \mid |x| \leq 1\}$, this is a *k-cell.* A (finite dimensional) *CW-complex* is a space constructed by attaching k-cells to each other using the following procedure. In step 0, X_0 is just a union of 0-cells (i.e. a set of vertices). In step k, we attach k-cells to X_{k-1} as follows: Consider a k-cell D^k and a continuous map $\phi : \partial D^k \to X_{k-1}$. This identifies the boundary of D^k to a subset of X_{k-1}. Suppose one does this for every D_i^k with $i \in I_k$ a certain index set. We get the space $X_k := X_k \cup \bigcup_{i \in I_k} D_i^k$. After $n \in \mathbb{N}$ steps, the resulting space $X := \bigcup_{k \geq 0} X_k$ is a CW-complex. The set $X^{(k)}$ is the subset of l-cells contained in X with $l \leq k$ (also called the *k-skeleton*).

Definition 2.1.10. Let X and Y be two CW-complexes, then a continuous map $q : X \to Y$ is *cellular* if

$$q\left(X^{(n)}\right) \subseteq Y^{(n)} \quad \text{for all } n.$$

Note that it is also allowed that $q\left(X^{(n)}\right) \subseteq Y^{(k)} \subseteq Y^{(n)}$ for $k \leq n$.

2.2 Cube complexes

Cube complexes are useful in the study complexes associated to right-angled Coxeter and Artin groups, as well as for right-angled buildings. We first define a general notion of a polytope.

Definition 2.2.1 ([24, Definition A.1.1])**.** A *convex polytope* or *convex cell* in $\mathbb{R}^n$ is the convex hull of a finite set of points. Its dimension is the dimension of the space it spans.

Definition 2.2.2 ([24, Section A.1])**.** Let C be a convex polytope in $\mathbb{R}^n$.

(i) A *supporting hyperplane* of C is hyperplane H of $\mathbb{R}^n$ such that $H \cap C \neq \emptyset$ and C is contained in the closure of one of the two half-spaces bounded by H.

(ii) A *face* F of C is a convex polytope that is formed by an intersection $H \cap C$ with a supporting hyperplane H and C. We denote the set of faces of C by Face($\mathbb{C}$).

(iii) The *barycentric subdivision* of a convex polytope C is the abstract simplicial complex $\operatorname{Flag}\big(\operatorname{Face}(C)\big)$.

(iv) A *convex cell complex* is a collection Ω of convex polytopes (also called cells) such that:

(1) Every face of every cell of Ω is itself a cell, i.e.

$$(\forall C \in \Omega)(\forall F \in \operatorname{Face}(C))(F \in \Omega);$$

(2) Every two cells of Ω are either disjoint or their intersection is a face of both cells, i.e.

$$(\forall C, C' \in \Omega)\big(C \cap C' = \emptyset \vee C \cap C' \in \mathrm{Face}(C) \cap \mathrm{Face}(C')\big).$$

The elements of Ω are called *cells*.

Definition 2.2.3 (Cube complex [46, Section 1.1])**.** A *cube complex* is a space built from cubes of arbitrary dimension glued together along their faces, i.e. a convex cell complex constructed from a union of Euclidean unit cubes $[0,1]^n \subseteq \mathbb{R}^n$. Moreover, Let X be a cube complex then we define

(i) The *induced metric* on X is the metric constructed from the piecewise metric on every k-cube.

(ii) A subcomplex of X is a cubecomplex formed by a subset of cubes of X glued together in the same way.

(iii) The *k-skeleton* of a cube complex is the subcomplex formed by forgetting all n-cubes with $n > k$.

Definition 2.2.4 (Link of a cube complex)**.** Let X be a cube complex and let $v \in X$ be a vertex. The *link* of v in X, denoted by $Lk(v, X)$, is the complex defined as follows:

- **Vertices:** Each edge (1-cell) containing v corresponds to a vertex in $Lk(v, X)$.
- **Simplices:** Every time there exists a $(k+1)$-dimensional cube in X containing v, then the k vertices that correspond to the k edges of this cube that contain v spans a k-simplex in $Lk(v, X)$.

In general this does not have to be a simplicial complex. It could be that there are more then one k-simplex between the same set of points (See Figure 2.3.2 as example).

Intuitively the link of a vertex v, is a complex, which shows the local structure of cells that contains v. For just graphs this is even simpler:

Definition 2.2.5. Let Γ be a graph and $v \in V(\Gamma)$ a vertex. Then we denote $Lk(v) := \big\langle w \in V(\Gamma) \mid w, v \in E(\Gamma)\big\rangle$ to be the subgraph induced by its adjacent vertices in Γ, and $St(v) := \big\langle Lk(v) \cup \{v\}\big\rangle_\Gamma$ to be the subgraph induced by v and its neighbors.

2.3 CAT(0) cube complexes

It will be the case that both the Davis complex and the Deligne complex (that we will introduce in Section 2.7 and Section 2.10) of Right-angled Artin groups are CAT(0) cube complexes. For a more in-depth look at CAT(0)-cube complexes we refer to the book [46] of CAT(0) cube complexes by Schwer. We first define what a geodesic metric space is.

Definition 2.3.1 ([36, Definition 5.3.1])**.** Let (X, d_X) be a metric space and $x, y \in X$ be two points.

(i) A *geodesic* τ between x and y is a isometric embedding $\tau : [0, L] \to X$ from the closed interval $[0, L] \subseteq \mathbb{R}$ to X, i.e.

$$\big(\forall t, t' \in [0, L]\big) \Big(|t - t'| = d_X\left(\tau(t), \tau(t')\right)\Big),$$

such that $\tau(0) = x$, $\tau(L) = y$.

(ii) The space (X, d_X) is called a *geodesic metric space* or *geodesic space* if for every pair $x, y \in X$ there exists a geodesic τ between x and y.

(iii) Let $x, y, z \in X$ be three points such that for each pair there is a geodesic $\tau_{x,y}, \tau_{y,z}$ and $\tau_{z,x}$ respectively. Then $\tau_{x,y}([0, 1]) \cup \tau_{y,z}([0, 1]) \cup \tau_{z,x}([0, 1]) =: \Delta_{x,y,z}$ is called a *geodesic triangle*.

Definition 2.3.2. Let $\Delta := \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} \in [(0, 0), (1, 0)] \cup [(0, 0), (1/2, \sqrt{3/4})] \cup [(1/2, \sqrt{3/4}), (1, 0)]\}$. This is just a triangle in $\mathbb{R}^2$ with length 1 for every edge[3]. In the case of Definition 2.3.1 (iii), suppose $\Delta_{x,y,z}$ is a geodesic triangle in X, then there is a natural bijective continuous map $\pi : \Delta \to \Delta_{x,y,z}$, such that

$$\begin{aligned}
\pi : [(0, 0), (1, 0)] &\xrightarrow{\sim} [0, 1] \xrightarrow{\tau_{x,y}} X;\\
\pi : [(1, 0), (1/2, \sqrt{3/4})] &\xrightarrow{\sim} [0, 1] \xrightarrow{\tau_{y,z}} X;\\
\pi : [(1/2, \sqrt{3/4}), (0, 0)] &\xrightarrow{\sim} [0, 1] \xrightarrow{\tau_{z,x}} X.
\end{aligned}$$

Definition 2.3.3 (CAT(0) space)**.** Let (X, d) be a geodesic space. Then X is CAT(0) if for all $x, y, z \in X$ and $\Delta_{x,y,z}$ a geodesic triangle, we have

$$d_{\mathbb{R}^2}(p, q) \geq d_X(\pi(p), \pi(q)) \quad \forall p, q \in \Delta^2. \tag{2.1}$$

If X is only locally CAT(0) (i.e. for every point there is a neighborhood of this point such that the relative topology in this neighborhood is CAT(0)), then we call X *non-positively curved*.

Definition 2.3.4 (Proper metric space)**.** A metric space is *proper* if sets are compact if and only if they are bounded and closed.

In this thesis our metric spaces will always be with *proper* metric spaces.

Remark 2.3.5. The geodesic triangle between three points does not need to be unique, so in Definition 2.3.3 the inequality (2.1) needs to be satisfied for every possible geodesic triangle between three points. Intuitively, one can have

[3]Here $[(a, b), (c, d)]$ is just the line segment $\{t \cdot (a, b) + (1 - t) \cdot (c, d) \mid t \in [0, 1]\}$.

the following picture in mind: “triangles in our CAT(0) space are squashed compared to triangles in $\mathbb{R}^2$”.

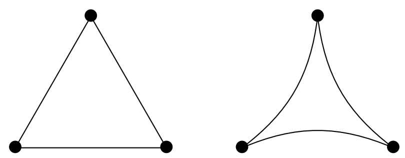


Figure 2.3.1: Comparing geodesic triangle in $\mathbb{R}^2$ with those in a CAT(0) space.

Example 2.3.6. (i) Let T be a tree with metric d the path metric, then (T, d) is clearly a CAT(0)-space.

(ii) Consider the sphere $\mathcal{S}^2 \subseteq \mathbb{R}^3$, this space is not CAT(0); moreover, in this space, for every nontrivial geodesic triangle, the inequality (2.1) is reversed.

We will give some properties of CAT(0) spaces. Later the point will be to prove that complexes on which for example Artin groups act geometrically will be CAT(0). Spaces that are CAT(0) have a nice structure (Theorem 2.3.7, Lemma 2.3.17), and this we will use for the $K(\pi, 1)$ conjecture and for quasi-isometric classifications of right-angled Artin groups.

Theorem 2.3.7 ([24, Theorem I.2.6])**.** *A complete* CAT(0)*-space is contactable*[4].

Proof. See [46, Proposition 3.8 (2)]. □

Hence, these spaces are always simply connected.

Definition 2.3.8. A CAT(0)-cube complex is a cube complex which is CAT(0) for the induced metric on the cube complex (see Definition 2.2.3 (i)).

The following easy criterion to be a locally CAT(0) cube complex we will often use (for example in Theorem 3.3.10 and Theorem 5.2.12, to prove that a right-angled Salvetti complex is non-positively curved).

Theorem 2.3.9 (Gromov's Link Condition for Cube Complexes [46, Theorem 4.43])**.** *A finite-dimensional cube complex is non-positively curved if and only if the link of every vertex is a flag simplicial complex.*

Theorem 2.3.10 (Gromov [35, Definition 34 & Theorem 41])**.** *A simply connected cube complex is* CAT(0) *if and only if the link of every vertex is a flag simplicial complex.*

Theorem 2.3.11 ([35, Theorem 31])**.** *A cube complex is complete if and only if there is no infinite ascending chain of cubes.*

[4]There exist a deformation retract to a point.

Example 2.3.12. The fact that in Theorem 2.3.9 the link needs to be a simplicial complex, prohibits cube complexes as:

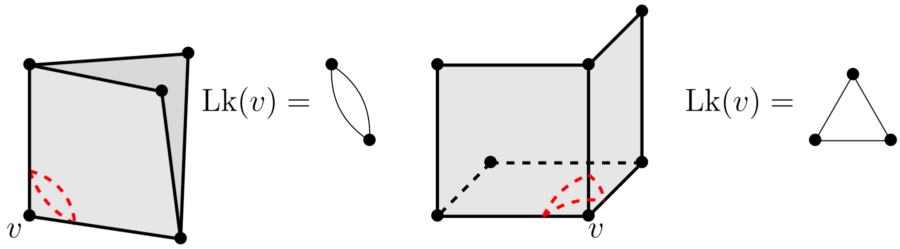


Figure 2.3.2: Non-CAT(0) cube complexes.

The first cube complex of Figure 2.3.2 consists of two 2-cubes which are attached by two of their 1-faces. The link of the vertex v consists of two edges that are two times connected by a 1-simplex. However, this is not a simplicial complex because the intersection between simplices here is not a simplex. In the second cube complex, the link of v is a simplicial complex but not a flag complex (the triangle is not "filled in").

Proposition 2.3.13 ([46, Proposition 3.11])**.** *Let X be a complete* CAT(0) *space and let $F \subseteq X$ be a closed convex subset (in particular, a face of a cube complex). Then for each $x \in X$, there is a unique point denoted by $proj_F(x) \in F$ such that*

$$d_X(x, proj_F(x)) = \inf_{y \in F}\{d_X(x, y)\}.$$

We call this element the *projection* of x on F.

Definition 2.3.14. Let X be a CAT(0) cube complex and $C, \tilde{C} \subseteq X$ two closed convex subsets. Then C and $\tilde{C}$ are parallel if

$$proj_C(\tilde{C}) = C \text{ and } proj_{\tilde{C}}(C) = \tilde{C}.$$

Property 2.3.15 ([33, Section 3.1])**.** *Let X be a* CAT(0) *cube complex and let $e, \tilde{e}$ be two 1-cubes (i.e. edges). Then e and $\tilde{e}$ are parallel if and only if there exist a sequence $e = e_0, e_1, \dots, e_n = \tilde{e}$ of 1-cubes such e_i is opposite e_{i+1} in a 2-cube.*

Definition 2.3.16 ([16, Section 2.1])**.** A cellular map $q : X \to Y$ between two CAT(0) cube complexes is *cubical* if for every cube $K \subseteq X$, it can be decomposed as

$$q : K \to \tilde{K} \xrightarrow{\sim} q(K) \subseteq Y,$$

where $K \to \tilde{K}$ is a projection map from K to a face $\tilde{K}$ of K in X and $q|_{\tilde{K}}$ is an isometry to $q(K)$.

Lemma 2.3.17 ([33, Lemma 4.5 & Lemma 4.6])**.** *Let $q : X \to Y$ be a cubical map between two* CAT(0) *cube complexes. Then the following hold.*

(i) *If $\sigma_1 \subseteq \sigma_2$ are cubes in Y and $y_i \in \sigma_i$ are interior points, then there is a natural embedding $q^{-1}(y_2) \hookrightarrow q^{-1}(y_1)$. This embedding is compatible with composition and inclusion.*

(ii) *For every convex set $A \subseteq Y$, every connected component of $f^{-1}(A)$ is convex.*

2.4 Cayley 2-complex

We need one last definition before going the Complexes associated to Coxeter and Artin groups. Is the Cayley 2-complex, that is a generalization of the Cayleygraph.

Definition 2.4.1 ([24, Section 2.2])**.** Let G be a finitely generated group with generating set S. A *Cayley 2-complex* $Cay_2(G)$ for G is any two-dimensional cell complex that is simply connected such that G acts on $Cay_2(G)$, and on the vertex set this action is transitive and free.

The 1-skeleton of a Cayley 2-complex for G will always be the Cayley graph of the group G for some generating set of G (See [24, Theorem 2.1.1]).

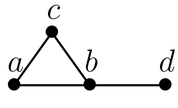
2.5 The nerve of a Coxeter group

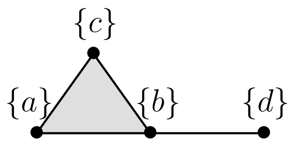
The first complex we define of a Coxeter group W_Γ is not much different than the defining graph Γ.

Definition 2.5.1 (The nerve of a Coxeter group, [24, Section 7.1])**.** Let W_Γ be a Coxeter group. The poset $\mathcal{S}_\Gamma^f \setminus \emptyset$ (where $\mathcal{S}_\Gamma^f := \{T \mid T \subseteq V(\Gamma)$ & W_T spherical$\}$) (by inclusion) is an abstract simplicial complex (one trivially checks the axioms of Definition 2.1.1) which is called the *nerve* of W_Γ.

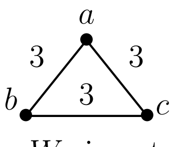
Intuitively the nerve of a Coxeter group with defining graph g is the simplicial complex made out of Γ by attaching triangles if possible and if this clique forms a spherical subgroup.

Example 2.5.2. (i) Let Γ be a graph without cycles of length 3, then the nerve will be just Γ.

(ii) For $\Gamma =$ 

we have nerve(Γ) $=$ 

.

(iii) If $\Gamma :=$ 

, then the nerve is just the graph Γ, without it "filled in" since W_Γ is not spherical.

Remark 2.5.3. Clearly a subset $T \subseteq V(\Gamma)$ is spherical (i.e. $T \in \mathcal{S}_\Gamma^f$) if and only if it spans a simplex in nerve(Γ).

The nerve of a Coxeter group often appears, such as when considering the link of vertices of the Coxeter cell (Theorem 2.8.4), it is used in an alternative way of constructing the exploded Salvetti complex (Remark 5.5.18) and it is also important in a classification of one ended Coxeter groups ([24, Theorem 8.7.2]). However, an equally important object is $\mathrm{Geom}\left(\mathrm{Flag}(\mathcal{S}_\Gamma^f)\right)$, that we will discuss now.

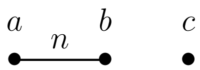
2.6 Fundamental domain

We will now define the fundamental domain, both the Davis complex and the Deligne complex (which we will see later in Section 2.7 and 2.10) are made up out of complexes isomorphic to the fundamental domain (see Remark 2.11.5). We will later also define the "real fundamental domain" (Definition 2.7.2), which is a subset of $\mathbb{R}^n$, however combinatorially these two notions will coincide.

Definition 2.6.1. Let W_Γ be a Coxeter group. Its fundamental domain is the following geometric realization of a simplicial complex, which we will denote by K_Γ

$$K_\Gamma := \mathrm{Geom}\left(\mathrm{flag}(\mathcal{S}_\Gamma^f)\right) \text{ where } \mathcal{S}_\Gamma^f := \{T \mid T \subseteq S \ \&\ W_T \text{ spherical}\}.$$

Remark 2.6.2. We will often see the fundamental domain as cube complex as follows. The cubes are the intervals $[T_1, T_2] := \{T' \in \mathcal{S}_\Gamma^f \mid T_2 \subseteq T' \subseteq T_2\}$ (the dimension of a cube then coincides with $|[T_1, T_2]| - 1$). This is equivalent with $\mathrm{Geom}\left(\mathrm{flag}(\mathcal{S}_\Gamma^f)\right)$ but forgetting the simplices that skips over a carnality (i.e. if σ is a simplex with $a = \min_{T \in \sigma^{(0)}}(|T|)$ en $b = \max_{T \in \sigma^{(0)}}(|T|)$, if there is a $t \in \{a, a+1, \ldots b\}$ such that there is no $T' \in \sigma^{(0)}$ with $|T'| = t$ then we forget about T).

Example 2.6.3. Let $\Gamma :=$ 

with $n \in \mathbb{N}_{>1}$. Then the fundamental domain is drawn in Figure 2.6.1.

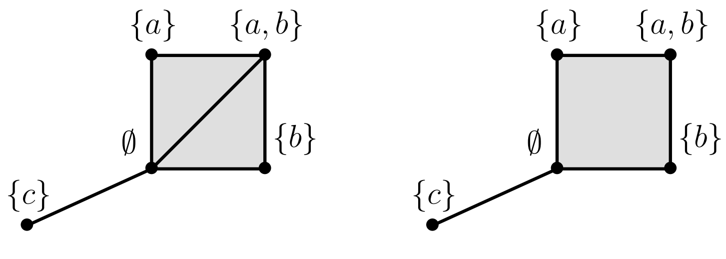


Figure 2.6.1: The fundamental domain and its cubical representation.

2.7 Davis complex of a finite Coxeter group

One can construct the Davis complex C_Γ for finite Coxeter group in multiple ways. First, we will construct it in a geometric way such that C_Γ is a subset of $\mathbb{R}^n$. At this point one could look back at Definition 1.2.4, to remind oneself of the hyperplanes of the Tits- representation.

Definition 2.7.1 (W_Γ-permutahedron, Coxeter cell, [24, Definition 7.3.1])**.** Suppose W_Γ a finite Coxeter group. Consider $x \in \mathbb{R}^n$ (with $n := |V(\Gamma)|$) such that the distance to every fixed hyperplane (the fixed hyperplanes of the Tits representation) is 1. Then the *Coxeter cell* is the convex cell obtained by taking the convex closure of the set $x^{W_\Gamma} =: C_\Gamma$ (i.e. the orbit of x, where we identify W_Γ with its Tits representation).

The following definition is some literature [39] also just called the fundamental domain. However, we will see that these objects coincide (at least combinatorially).

Definition 2.7.2 (Real fundamental domain)**.** The *real fundamental domain* is the intersection with the interior of the fundamental cone (see Definition 1.2.10) and the Coxeter cell.

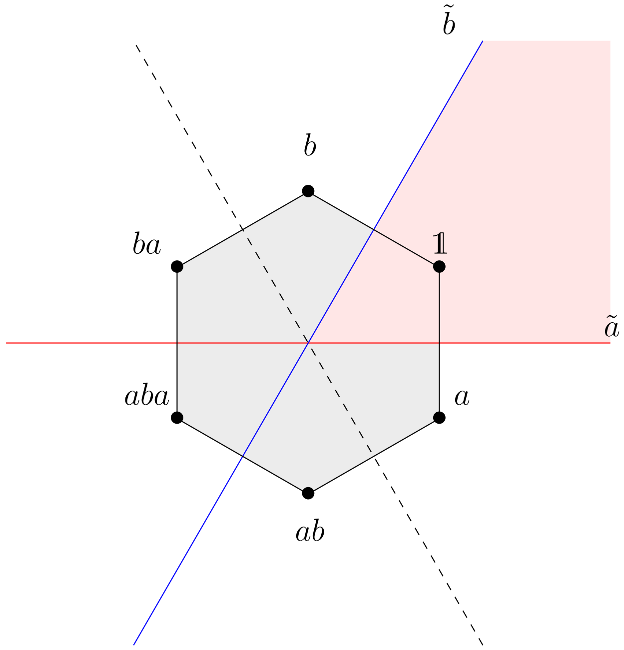


Figure 2.7.1: Coxeter cell of W_Γ.

Example 2.7.3. Consider W_Γ the Coxeter group with defining graph $\Gamma :=$ a —3— b . Then the Coxeter cell is drawn in Figure 2.7.1. Which is a convex hull of a hexagon. The points of C_Γ are the elements of the orbit x^{W_Γ}. Between two points is an edge if and only if there is an element $s \in V(\Gamma)$ of the generating set of W_Γ such that these two points are mapped to each other by this element s. The fundamental cone is drawn in red. The real fundamental domain (here a quadrangle) is the intersection with this hexagon.

Remark 2.7.4. Consider C_Γ the Coxeter cell of a finite Coxeter group of type Γ.

(i) It is clear that the 1-skeleton on the Coxeter cell is the Cayleygraph of the associated Coxeter group, because there is a bijection between the vertices of C_Γ and the orbit of a vertex in C_Γ by the group W_Γ, which is in bijection with the elements of W_Γ (by Theorem 1.2.7).

(ii) One can verify that at least combinatorially the real fundamental domain and the fundamental domain coincide. Moreover the barycentric subdivision of C_Γ exists of fundamental domains glued together (see later Remark 2.11.5 and Example 2.11.6).

(iii) The term "Permutahedron" comes from the fact that the Coxeter cell of the Coxeter group Sym_n is a Permutahedron in a classical sense (a real polytope in dimension $n-1$).

(iv) In Example 2.7.3, there were only points, edges and one plane. For higher-dimensional cases (in case $|V(\Gamma)| > 2$), one will get more planes or arbitrary n-cells. Such an n-cell corresponds uniquely to a coset of a special subgroup of rank n. In Figure 2.7.1, the edge from $\mathbb{1}$ to a corresponds to the coset $W_{\{a\}}$ and the edge from ab to aba to $abW_{\{a\}}$ and so forth. The whole plane corresponds to $W_{\{a,b\}}$ (We will see this more generally in Theorem 2.7.5).

(v) The name "(real) fundamental domain" comes from the fact that the action of W_Γ on the 1-skeleton of C_Γ (by left multiplication) induces an action $W_\Gamma \curvearrowright C_\Gamma$ on the full Coxeter cell, for which the orbit of the fundamental domain is the complete complex C_Γ.

The following Theorem serves also as a basis for how we later will define the Deligne complex. It also gives us a consisted way in defining the fundamental domain, Davis complex and the Deligne complex.

Theorem 2.7.5 ([24, Lemma 7.3.3])**.** *For W_Γ a finite Coxeter group and* $\mathrm{Face}(C_{W_\Gamma})$ *the poset of faces of the Coxeter cell. The map $W_\Gamma \to V(C_{W_\Gamma}) : w \mapsto x^w$ induces a poset isomorphism $W_\Gamma\mathcal{S}^f \xrightarrow{\sim} \mathrm{Face}(C_{W_\Gamma})$.*

Remark 2.7.6 (Davis complex seen as an abstract simplicial complex)**.** Suppose W_Γ a Coxeter group and let C_{W_Γ} be the Davis complex. The poset of spherical cosets $W_\Gamma\mathcal{S}^f$ and the poset of faces of the Davis complex $\mathrm{Face}(C_{W_\Gamma})$ are by Theorem 2.7.5 isomorphic, we thus will sometimes refer to the Davis complex as $W_\Gamma\mathcal{S}^f$. We will refer to the fundamental domain as the poset $\mathcal{S}^f$, later we will define the Deligne complex as $A_\Gamma\mathcal{S}^f$.

One can verify that The geometric realization of $\mathrm{Flag}(W_\Gamma\mathcal{S}^f)$ is isomorphic to the barycentric subdivision of C_{W_Γ} (see Definition 2.2.2 (iii)). As an example in

Figure 4.2.3 is Geom$\big(\mathrm{Flag}(W_\Gamma \mathcal{S}^f)\big)$ shown for $\Gamma := \overset{a}{\bullet}\overset{3}{\rule[0.5ex]{3em}{0.4pt}}\overset{b}{\bullet}$.

2.8 Davis complex of an infinite Coxeter group

As in Remark 2.7.6 one can define the Davis complex for arbitrary Coxeter groups just as $W_\Gamma \mathcal{S}^f$. Despite that, we will construct a way such that we can view this as a complex (a complex that would coincide with Coxeter cell C_Γ in the finite case) one can use the following construction.

Construction 2.8.1. Let W_Γ be an arbitrary Coxeter group. Let C_Γ the following complex: start with the Cayleygraph $Cay(W_\Gamma)$. If W_Λ is a spherical subgroup; then $Cay(W_\Lambda)$ is a subgraph of $Cay(W_\Gamma)$, also the 1-skeleton of the Coxeter cell (from definition 2.7.1) C_Λ coincides with $Cay(W_\Lambda)$. Such that there is natural way to attach the Coxeter cell C_Λ to $Cay(C_\Gamma)$. If there is a spherical subgroup $W_\Pi \leq W_\Lambda \leq W_\Gamma$; then is C_Π a face of C_Λ such that if one attaches C_Π to $Cay(W_\Gamma)$, this becomes a subcomplex of C_Λ in C_Γ. Similarly, one attaches C_Λ to every coset wW_Λ inside $Cay(W_\Gamma)$.

Definition 2.8.2. Let W_Γ be an arbitrary Coxeter group. The resulting cell complex of Construction 2.8.1 is called the *Davis complex* (also called Coxeter complex) of W_Γ. Just as for the finite case we can define a *real fundamental domain* for the infinite case. By considering the union of the fundamental domains of each $C_\Lambda \subseteq C_\Gamma$ for each spherical subgroup $W_\Lambda \leq W_\Gamma$ (here not for the cosets gW_Λ with $g \notin W_\Lambda$).

Example 2.8.3. (i) We construct the Davis complex of type $\Gamma := \overset{b}{\bullet}\overset{3}{\rule[0.5ex]{3em}{0.4pt}}\overset{a}{\bullet}\overset{2}{\rule[0.5ex]{3em}{0.4pt}}\overset{c}{\bullet}$.
We first construct its Cayleygraph. Then we look a the spherical cosets of W_Γ, on these subgraphs we will attach the associated Coxeter cells. For example the spherical coset $\mathbb{1}W_{\{a,b\}}$ has an associated Coxeter cell drawn in Figure 2.7.1. Doing the same to all the cosets of the form $gW_\emptyset, gW_{\{a\}}, gW_{\{b\}}$, $gW_{\{c\}}, gW_{\{a,b\}}, gW_{\{b,c\}}$ for $g \in W_\Gamma$ we get the cell complex in Figure 2.8.1.

In Figure 2.8.1, the red colored part is the fundamental domain, which is in this case isomorphic with two quadrangles glued together at one side. The orbit of the fundamental domain is the complete complex. There are 6 Coxeter cells that contain the vertex a, they correspond to the cosets $aW_\varnothing, W_{\{a\}}, aW_{\{b\}}$, $aW_{\{c\}}, W_{\{a,b\}}$ and $W_{\{a,c\}} \in W_\Gamma \mathcal{S}^f$.

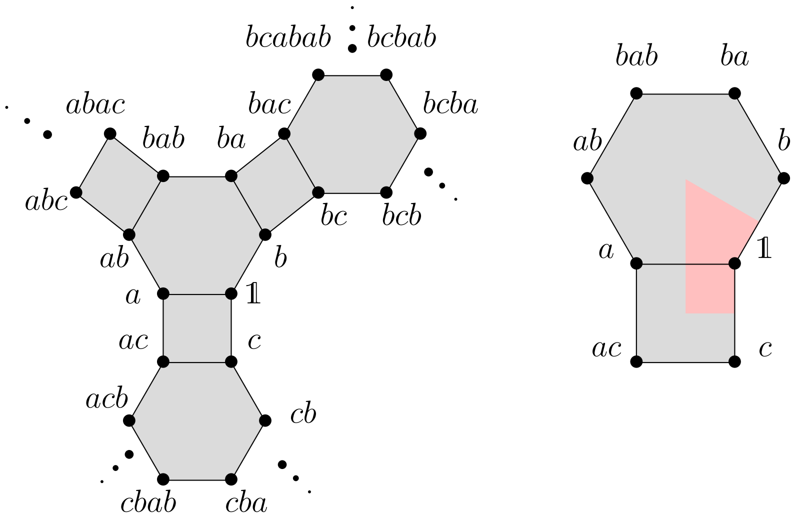


Figure 2.8.1: Left: the Davis complex, right: Coxeter cells containing 𝟙.

(ii) Let $\Gamma :=$ $\overset{c}{\bullet}$ $\quad \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$, the Davis complex is drawn in Figure 2.8.2 it is drawn in a hyperbolic projection. In this way we can represent the 0-skeleton of this complex as the orbit of one vertex by the reflections of lines. The two lines that intersect perpendicular correspond with a and b since they commute. The generator c does not commute with anything, hence, c correspond with the line that only intersect a and b at infinity.

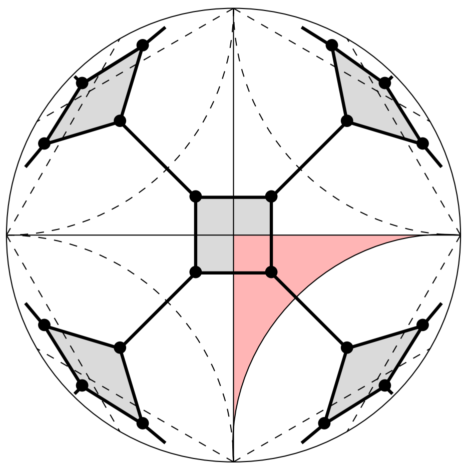


Figure 2.8.2: Hyperbolic projection of the Davis complex.

In Example 2.6.3 we already constructed the fundamental domain of Γ in Figure 2.6.1. One can see this again as the intersection between the Davis complex and the red region (that sometimes is also called the fundamental domain). Also if we look at the barycentric subdivision, the fundamental domain is a subcomplex of this barycentric subdivision (also see Figure 2.11.1).

We will not prove the following theorem, however one should check this theorem on an example to get some intuition in these complexes.

Theorem 2.8.4 ([24, Proposition 7.3.4.])**.** *Let C_Γ be the Coxeter complex constructed in Construction 2.8.1. Then it satisfies the following:*

(i) *The 1-skeleton of C_Γ is the Cayleygraph of W_Γ, and the 2-skeleton of C_Γ is the Cayley 2-complex;*

(ii) *The link of every vertex of C_Γ is isomorphic to the nerve(Γ) of W_Γ;*

(iii) *A subset of W_Γ forms a cell in C_Γ if and only if it corresponds with a spherical coset;*

(iv) *The poset of cells of C_Γ is isomorphic to the poset $W_\Gamma \mathcal{S}^f$.*

Definition 2.8.5. Let W_Γ a Coxeter group and C_Γ its Davis complex, then there is a natural faithful action by W_Γ on C_Γ by left multiplication. Take $w \in W_\Gamma$ the action of the element sends the Coxeter cell corresponding the the coset gW_Λ to wgW_Λ.

Remark 2.8.6. (i) For the finite case Definition 2.7.1 and Definition 2.8.2 coincides, precisely because W_Γ is itself spherical.

(ii) When we say Coxeter polytope or Coxeter Cell instead of Coxeter complex/ Davis complex we explicit focus on the fact that it comes from a finite Coxeter group.

(iii) Theorem 2.8.4 (ii) and (iii) tells us that every vertex is contained in exactly one Coxeter cell for each spherical subgroup of W_Γ.

(iv) If W_Γ is a right-angled Coxeter group, then the Davis complex is clearly a cube complex, since the Coxeter cells are all cubes.

Theorem 2.8.4 immediately implies the following, now also for infinite Coxeter groups.

Corollary 2.8.7. *The barycentric subdivision of the Davis complex is isomorphic with* $\text{Geom}\left(\text{Flag}(W_\Gamma \mathcal{S}^f)\right)$.

We can define a metric on the Davis complex. The obtained metric space will turn out to be CAT(0).

Definition 2.8.8 (Moussong metric)**.** Let C_Γ be the Davis complex of a Coxeter group W_g. The *Moussong Metric* on C_Γ is the metric induced by the piecewise euclidean metric in every Coxeter cell of C_Γ.

Remark 2.8.9. The Moussong metric can also be constructed as follows. Consider the real fundamental domain (which we will denote by K_Γ) (Definition 2.7.2) with its induced metric in $\mathbb{R}^n$ (as a subspace). If Γ is not spherical we can glue each real fundamental domain of spherical subgroups in a natural manner (as in definition 2.8.2). Hence, we have metric on this space (this space is also called the *Coxeter block of type* Γ in [24, page 337]). We then extend this metric to the whole Davis complex C_Γ by the action of W_Γ on C_Γ, this is possible since $K_\Gamma^{W_\Gamma}$ tiles the whole Davis complex. Suppose now a right-angled case Γ. The Coxeter cells (for Λ a complete subgraph of Γ) are always cubes ($= [-1,1]^n$ with n the rank of the Coxeter group W_Λ), and the real fundamental domain of a Coxeter cell is the intersection of this cube with a cone bounded by orthogonal hyperplanes (hence a $[0,1]^n$ cube). The Coxeter block is then cubes glued together, and hence, a cube complex. More formally we do the following: The fundamental domain K_Γ is isomorphic to $\{C \in W_\Gamma \mathcal{S}^f \mid \mathbb{1} \in C\}$ by

$$\mathcal{S}^f \to \{C \in W_\Gamma \mathcal{S}^f \mid \mathbb{1} \in C\} : T \mapsto A_T.$$

There is an induced metric on K_Γ in $\mathbb{R}^n$, we copy this metric to $\{C \in W_\Gamma \mathcal{S}^f \mid \mathbb{1} \in C\}$ and extend this metric to the whole Davis complex since $C_{W_\Gamma} = \{C \in W_\Gamma \mathcal{S}^f \mid \mathbb{1} \in C\}^{A_\Gamma}$ (the action by left multiplication).

This alternative construction of the Moussong metric will be interesting to define a metric on buildings of type Γ later in Chapter 4.

Theorem 2.8.10 ([39, Theorem 3.5])**.** *For every Coxeter group W_Γ is the Davis complex C_Γ a* CAT(0) *space for the Moussong metric (Definition 2.8.8). Moreover, the action of W_Γ in the Davis complex is geometric*[5].

Proof. See [24, Chapter Twelve]. It was first proven by Moussong in his Master's thesis in 1988 who was a student of Michael Davis. □

Theorem 2.8.11 ([18, Theorem 3.5])**.** *For W_Γ a right-angled Coxeter group then the Davis complex C_Γ is a* CAT(0) *cube complex.*

2.9 Salvetti complex

The next space we will discuss will be a metric space such that the fundamental group is a predetermined Artin group. The Salvetti complex will be interesting for the $K(\pi, 1)$ conjecture. In addition, this complex will be of great use to find quasi-isometries between right-angled Artin groups. The universal cover of this space will be similar to the Deligne complex. We will first define the Salvetti complex for right-angled Coxeter groups.

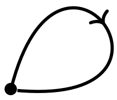
Definition 2.9.1. Consider Γ a simple graph and a family of pointed topological spaces[6] $\{(X_v, p_v)\}_{v \in V(\Gamma)}$ with index set $V(\Gamma)$. Then the Γ-*graph product* is a topological space $\widetilde{\prod}_{v \in V(\Gamma)}(X_v, p_v)$ defined as

$$\widetilde{\prod}_{v \in V(\Gamma)}(X_v, p_v) := \bigcup_{\substack{\Delta \subseteq \Gamma \\ \text{clique}}} \left(\prod_{v \notin V(\Delta)} \{p_v\} \times \prod_{v \in V(\Delta)} X_v \right) \subseteq \prod_{v \in V(\Gamma)} X_v.$$

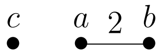
[5] see Definition 6.2.1

[6] A pointed topological space (X, p) is a topological space X with a chosen basepoint $p \in X$.

Definition 2.9.2 (Salvetti complex for RACG)**.** The *Salvetti complex* $\mathcal{S}_\Gamma$ of a right-angled Coxeter group W_Γ is the Γ-graph product constructed if for every $v \in V(\Gamma)$ we use $(X_v, p_v) := (S^1, (0,1))$ in Definition 2.9.1 (where S^1 is the unit circle). We will write S^1_v and $\bullet_v = p_v$ if we want to denote the unit circle and basepoint associated for the index $v \in V(\Gamma)$. Hence,

$$\mathcal{S}_\Gamma := \widetilde{\prod}_{v\in V(\Gamma)} (S^1_v, \bullet_v), \quad \text{where } S^1 = $$  .

Remark 2.9.3. Intuitively, one can see the Salvetti complex of W_Γ as the complex starting from a wedge of $n := |V(\Gamma)|$ circles, if two vertices are connected we attach a torus($\cong S^1 \times S^1$) to this. If a set of k vertices form a clique we attach a k-torus($\cong \bigtimes_{1\le i\le k} S^1$) to this complex. The resulting complex is then our Salvetti complex.

Example 2.9.4. Consider the simple graph $\Gamma :=$ 

. The Salvetti complex $\mathcal{S}_\Gamma$ is draw in Figure 2.9.1. The fundamental group of this space is $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$, that is also the Artin group $A_\Gamma = \langle a, b, c \mid ab = ba \rangle$. This will also true in general; see Theorem 2.9.17.

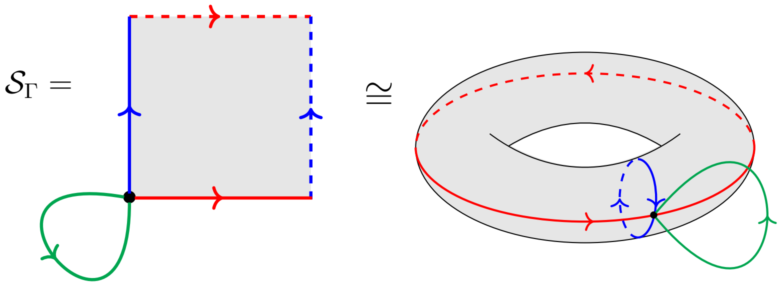


Figure 2.9.1: The Salvetti complex of Γ.

Remark 2.9.5. As seen in Figure 2.9.1 we will often visualize a torus as square where we glue the two pair of opposite sides together, i.e. in Figure 2.9.1 we first glue the two red lines together then we get a cylinder, having this we glue the two opposite blue lines together and get a torus. See also Figure 2.9.2.

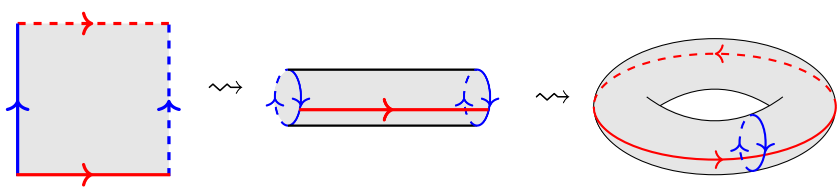


Figure 2.9.2: Visualization of a torus

For general non-right-angled Artin groups one cannot define the Salvetti complex using Definition 2.9.1, for this we need to construct it out of the Davis complex of W_Γ .

Definition 2.9.6 (Orientation of a Coxeter cell)**.** Let W_Γ be a finite Coxeter group and let C_Γ be the Coxeter cell from Definition 2.7.1. Suppose $w, w' \in V(C_\Gamma)$ are arbitrary vertices of C_Γ which are connected by an edge. An *orientation* of this edge points either from w to w' or from w' to w. For every vertex $v \in V(C_\Gamma)$ there is a unique [7] vertex v' directly opposite to v such that the vector $\vec{v}$ from v to v' passes through the center of C_Γ. *The oriented Coxeter cell* $\tilde{C}_\Gamma$ of C_Γ (depending on v) is the structure resulting when we give every edge in the 1-skeleton of C_Γ the orientation such that the inner product with $\vec{v}$ is positive (we also say that $\tilde{C}_\Gamma$ is the oriented Coxeter cell with *basepoint* v).

Remark 2.9.7. (i) If the Coxeter polytope C_Γ has n vertices; then it also has n possible orientations.

(ii) Note that such an orientation of an edge is always well-defined; it cannot be the case that an edge is "perpendicular" to the vector $\vec{v}$. Suppose this is the case for the edge (corresponding to a generator $s \in V(\Gamma)$) between w and w' in the Coxeter cell C_Γ, then there is a path $p \subseteq W_\Gamma$ (respectively $p' \subseteq W_\Gamma$) from v to w (respectively w') such that the length of both paths are equal. But because w and w' are connected, we have $w^{-1}w' = s \in V(\Gamma)$ such that $p^{-1}p' = s$. Therefore, $p^{-1}p's = \mathbb{1}$, which is a contradiction by Lemma 1.1.7.

Example 2.9.8. Consider the Coxeter group of type $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$. The Coxeter cell (Figure 2.7.1) has 6 possible orientations, one for each vertex. Figure 2.9.3 shows one possible orientation.

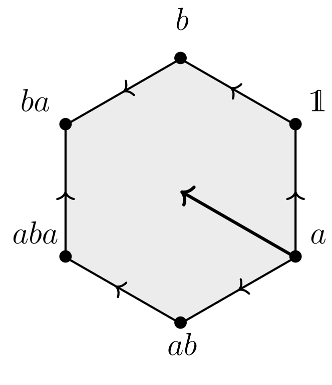


Figure 2.9.3: The oriented Coxeter cell with base point a.

Before we can define the Salvetti complex in general, we need the notion of the

[7]This is by picking a word g of maximal length in W_Γ (this exists since this group is finite). Then one can check that $v' := wv$ satisfies this condition (use Lemma 1.1.7).

Full Salvetti complex (In the paper by McCammond [39] it is just called the "Salvetti complex").

Construction 2.9.9. Let W_Γ an arbitrary Coxeter group and C_Γ its Davis complex. Consider a Coxeter cell C' which is a subcomplex of C_Γ (i.e. $C' \cong C_\Lambda$ for $\Lambda \in \mathcal{S}_\Gamma^f$) suppose this subcomplex has n possible orientations (as in Definition 2.9.6) call these $C'_1, C'_2, \ldots, C'_n$, then we replace C' with the n oriented Coxeter cells. Two oriented Coxeter cells C'_l and C'_k are connected by a smaller oriented Coxeter cell $C'' \subseteq C'_l, C'_k$ if and only if the orientation of C'' is compatible with that of C'_L and C'_k.

Definition 2.9.10 (Full Salvetti complex)**.** Let W_Γ an arbitrary Coxeter group, the resulting space from Construction 2.9.9 by replacing each Coxeter cell by as many oriented Coxeter cells as the number of vertices in this cell, is the *full Salvetti complex*. We denote this by $F\mathcal{S}_\Gamma$.

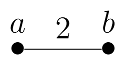
Remark 2.9.11. Let W_Γ a Coxeter group and $F\mathcal{S}_\Gamma$ the full Salvetti complex. By construction of $F\mathcal{S}_\Gamma$ there is a bijection between the points of $F\mathcal{S}_\Gamma$ and the points of C_Γ and thus also with the elements of W_Γ (see Remark 2.7.4 (i)).

To construct the Salvetti complex out of the full Salvetti complex, we will need to quotient out the action of W_Γ. Consider the following action.

Definition 2.9.12. Similar as in Definition 2.8.5 one can define an action of W_Γ on $F\mathcal{S}_\Gamma$. We first define this action on the 0-skeleton by left multiplication. Let $\tilde{C}$ be an oriented Coxeter cell in $F\mathcal{S}_\Gamma$ with base point $v \in W_\Gamma$, then $\tilde{C}^w$ (with $w \in W_\Gamma$) is the unique oriented Coxeter cell $\tilde{C}'$ with base point wv and which corresponds to the same unique spherical subgroup (see Remark 2.8.6(iii)), but now a different coset namely $gW_\Lambda \mapsto wgW_\Lambda$.

Definition 2.9.13 (Salvetti complex for arbitrary Coxeter groups)**.** Let W_Γ be an arbitrary Coxeter group let $F\mathcal{S}_\Gamma$ be the full Salvetti complex, the *Salvetti complex* of W_Γ is the quotient complex $F\mathcal{S}_\Gamma / W_\Gamma$ (by the action defined in Definition 2.9.12).

We will say that $\mathcal{S}_\Gamma$ is the Salvetti complex associated to W_Γ or to A_Γ. The Salvetti complex only depends on the defining graph Γ.

Example 2.9.14. (i) We first give an example in the finite case. Let $\Gamma :=$ 

, in the finite case the Coxeter complex only consists of one Coxeter cell C_Γ. The Full Salvetti complex has as many copies of this Coxeter cell as it has orientations. These oriented Coxeter cells are almost everywhere disjoint and only connect when they have an edge that has the same orientation. See Figure 2.9.4 for the construction of the full Salvetti complex of Γ. The full Salvetti complex $F\mathcal{S}_\Gamma$ consists of 4 squares, where each pair has either two mutual edges (that have the same orientation) or non.

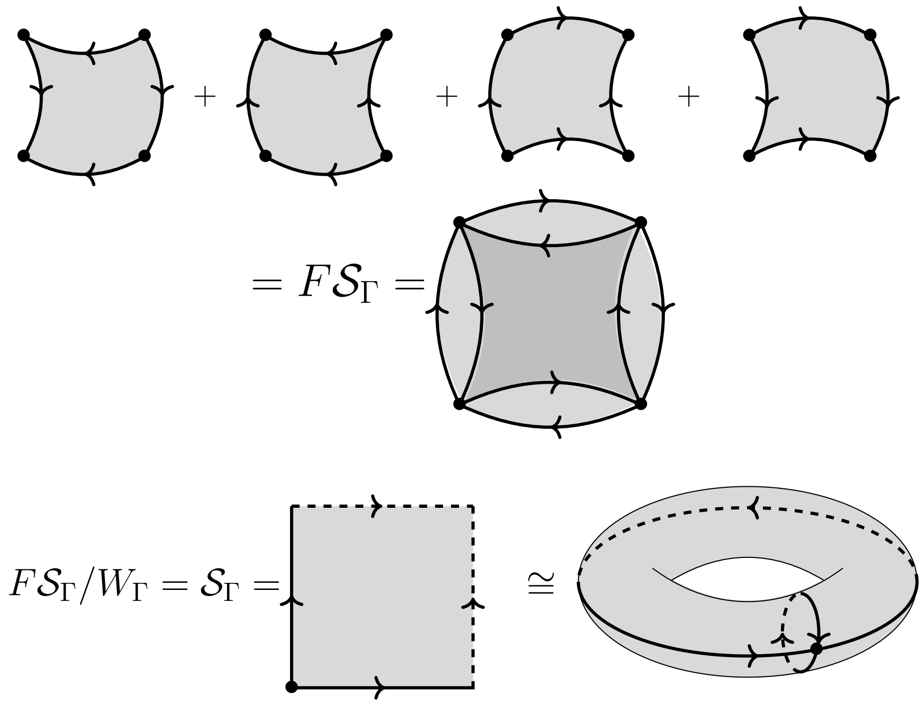
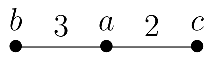


Figure 2.9.4: The full Salvetti complex $F\mathcal{S}_\Gamma$ and Salvetti complex $\mathcal{S}_\Gamma$ of Γ.

The Salvetti complex of Γ is then the quotient space (by the action of $W_\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$), we are then left with one square wherein we identify opposite edges. Hence, we have a torus, which has fundamental group $\mathbb{Z} \times \mathbb{Z}$.

(ii) For an infinite case consider $\Gamma :=$ 

. The full Salvetti complex and the Salvetti complex is drawn in Figure 2.9.5.

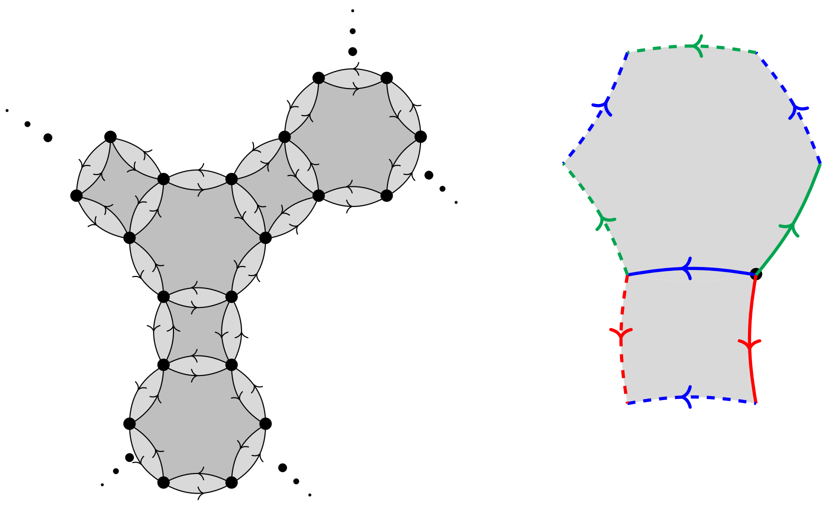


Figure 2.9.5: Left: the full Salvetti complex for Γ; right: the Salvetti complex.

The full Slaveti complex is the Davis complex (constructed in Example 2.8.3 (ii)) where we replace every Coxeter cell by oriented Coxeter cells. Different oriented Coxeter cells are glued together on sub-oriented Coxeter cells if these subcells have the same orientation. In Figure 2.9.5, is the Salvetti complex drawn. The blue loops and red loops span a torus (See Remark 2.9.5). In the hexagon spanned by the blue loop and green loop, we identify all the green edges (along their orientation), and the same for the blue edges.

Later we will see the Salvetti complex of right-angled Artin groups as cube complexes (See Definition 5.2.6), hence it is natural to call the end points of these edges vertices (and not "points").

Definition 2.9.15. The Salvetti complex contains only contains one "vertex" (as seen in Figure 2.9.5 and 2.9.4), which is the point corresponding to the orbit of an arbitrary vertex $\bullet^{W_\Gamma}$ or for the right-angled case the vertex corresponding to $\prod_{v\in V(\Gamma)}\{\bullet_v\}$ in Definition 2.9.2. We will call $\bullet^{W_\Gamma} =: \bullet$ also the basepoint of the Salvetti complex.

Lemma 2.9.16. *The two constructions Definition 2.9.2 and Definition 2.9.13 coincide for right-angled Artin groups.*

Proof. Exercise. □

Hence, the long awaited theorem.

Theorem 2.9.17 ([39, page 18])**.** *For a right-angled Artin group A_Γ and $\mathcal{S}_\Gamma$ the Salvetti complex associated to Γ, the fundamental group $\pi_1(\mathcal{S}_\Gamma)$ is isomorphic to A_Γ.*

Sketch of proof. We can find the isomorphism explicitly by mapping the generators of $\pi_1(\mathcal{S}_\Gamma)$ (i.e. the edges containing the base point (one of course first needs to prove that every path is homotopic to a sequence of edges containing the base point)) to the generators of A_Γ (choosing the one that corresponds to the generator in Γ that corresponds to this edge in the Davis complex). The rest is an exercise. □

Remark 2.9.18. (i) The Davis complex is simply connected, however the full Salvetti complex is never simply connected. For example if one starts in any vertex and picks an arbitrary edge that contains this vertex. If we consider the path starting at this point going over this edge and then going back via the inverse oriented edge, we get a path that is not homotopic to the trivial path. In Theorem 2.12.1 we will show that the fundamental group of this space is isomorphic to the kernel of the natural morphism $A_\Gamma \twoheadrightarrow W_\Gamma$.

(ii) There is a distinction between the Salvetti complex of A_Γ and the presentation complex of A_Γ. For a finitely generated group G the *presentation* *complex* is the 2-dimensional cell complex constructed as follows: Let S the generating set of G, with $n = |S|$. One start with a wedge of n circles connected at one point, for every word in the presentation we add a 2-cell and attach this word to the boundary of this 2-cell. For a RAAG the 2-skeleton of the Salvetti complex coincides with the presentation complex (see [9, Section 1.1]). However, the Salvetti complex could also contain higher dimensional cells. For example, let Γ be complete graph on three vertices (i.e. a triangle) then the Salvetti complex is a 3-dimensional cube where we identify parallel edges with each other and parallel faces. This complex will be a non-positively curved cube complex (See later proof (2) of Theorem 3.3.10). The presentation complex of A_Γ is the second cube complex drawn in Figure 2.3.2, where we identify parallel edge to parallel edges. Besides, this presentation complex is not a positively curved space.

2.10 Deligne complex

Some highlights of the correlation between the Coxeter groups and Artin groups are shown by Pierre Deligne (The first Belgian who obtained a Fields medal) in [25]. In this section, we will introduce the Deligne complex. A complex with similar properties as Davis complex for a Coxeter group, but now for an Artin group.

Remark 2.10.1. The Davis complex C_Γ was constructed from attaching Coxeter cells to the Cayleygraph, from which we obtained the interesting fact that the poset of faces was isomorphic to the poset of spherical cosets $W_\Gamma\mathcal{S}^f$. The Deligne complex will be defined in the reverse order.

Definition 2.10.2. Consider A_Γ an Artin group of type Γ and let $A_\Gamma\mathcal{S}^f$ be the poset of spherical cosets of A_Γ. The *Deligne complex* is the simplicial complex $\text{Flag}(A_\Gamma\mathcal{S}^f)$.

Construction 2.10.3. Let $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$ be a right-angled defining graph. Consider the following complex D_Γ; The points of D_Γ are the elements of Artin group (i.e. the rank 0 cosets in $A_\Gamma\mathcal{S}^f$). The lines of D_Γ are the sets of all vertices contained in a rank 1 coset. In contrast the Davis complex for which the rank 1 cosets only contained two elements (for example $gW_a = \{g, ga\}$). The rank 1 cosets of Artin groups contains infinitely many elements (for example $gA_a = \{g, ga, ga^{-1}, ga^2, ga^{-2}, \dots\}$). The "planes" are the sets of all vertices contained in a rank 2 cosets and so on. Here, the "faces" of D_Γ corresponds to the cosets of A_Γ. We will also call D_Γ the Deligne complex of type Γ. Figure 2.10.1 shows the construction of the Deligne complex D_Γ of type $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$.

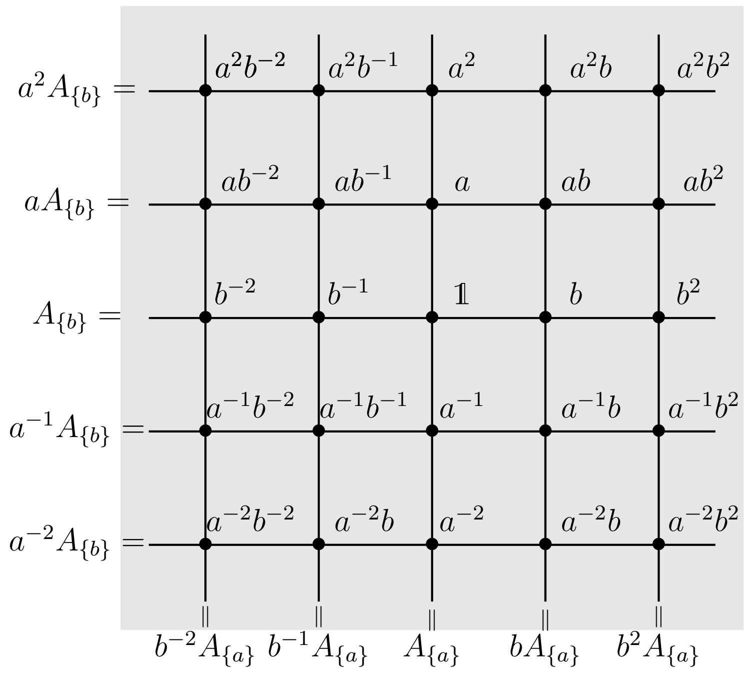


Figure 2.10.1: Deligne complex of Γ.

This space also coincides with the universal cover of the Salvetti complex of type $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$.

Remark 2.10.4. (i) One should look at Theorem 2.8.4 and Construction 2.10.3 and see the similarities.

(ii) In the Deligne complex D_Γ of type $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$, the coset $A_{\{a,b\}}$ clearly corresponds with a real plane. However in general, cosets of rank n of Artin groups that are not right-angled do not have a structure of $\mathbb{R}^{n+1}$.

For example consider D_Γ of type $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$, then the whole group is a spherical coset of rank 2. However, this coset will not span $\mathbb{R}^2$.

2.11 Modified Deligne complex and Davis complex

In this section we will give an alternative construction of the Davis complex and Deligne complex. For more information of this construction see [21, Section 1].

Construction 2.11.1. Let W_Γ be a Coxeter group with rank $n := |V(\Gamma)|$. Let Δ_n be a standard n-simplex (Definition 2.1.4) spanned by the elements of $V(\Gamma)$. Consider the following complex

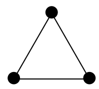
$$\hat{C}_\Gamma := W_\Gamma \times \Delta_n / \sim,$$

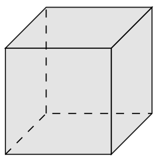
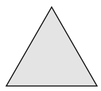
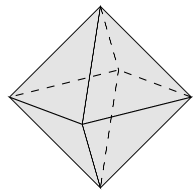
where $\sim$ is an equivalence relation $(w_1, x) \sim (w_2, y)$ iff $x = y$ and if there is a subsimplex $\delta_T \subseteq \Delta_n$ ($T \subseteq V(\Gamma)$) containing x in its interior such that $w_1^{-1}w_2 \in W_T$. Similarly on construct the complex

$$\hat{D_\Gamma} := A_\Gamma \times \Delta_n / \sim,$$

where $\sim$ is an equivalence relation $(a_1, x) \sim (a_2, y)$ iff $x = y$ and if there is a subsimplex $\delta_T \subseteq \Delta_n$ ($T \subseteq V(\Gamma)$) containing x in its interior such that $a_1^{-1}a_2 \in A_T$.

These complexes are also called the Davis complex and Deligne complex respectively in [21]. However, these complexes will not be the same as constructed in Section 2.7, it will be the dual of these complexes.

Example 2.11.2. Let $\Gamma :=$  , the Davis complex constructed as in Section 2.7 is a cube. The Davis complex constructed in Construction 2.11.1 is the dual of a cube.

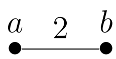
$C_\Gamma =$  $\Delta_n =$  $\hat{C}_\Gamma =$ 

The complex $\hat{C}_\Gamma$ is formed by attaching $|W_\Gamma|$ many n-simplices in such a way it satisfies "$\sim$".

Remark 2.11.3. In literature one regularly calls the complexes $\hat{C}_\Gamma$ and $\hat{D}_\Gamma$ discussed in Construction 2.11.1 the Davis and Deligne complex and the geometric realization of $\mathrm{Flag}(W_\Gamma\mathcal{S}^f)$ and $\mathrm{Flag}(A_\Gamma\mathcal{S}^f)$ are called the *modified Davis* and *modified Deligne complex* respectively.

Construction 2.11.4 ([20, page 8],[19])**.** We can give a natural cubical structure on the Deligne complex $\mathrm{Flag}(A_\Gamma\mathcal{S}^f)$. where we only connect vertices gA_T and $gA_{\tilde{T}}$ if $|T| + 1 = |\tilde{T}|$. The cubes are then of the form

$$[gA_{T_1}, gA_{T_2}] := \{gA_{\tilde{T}} \in A_\Gamma\mathcal{S}^f \mid T_1 \subseteq \tilde{T} \subseteq T_2\},$$

this would be a $\big|[gA_{T_1}, gA_{T_2}]\big|$-dimensional cube. Completely similar we can give a natural cubical structure on the Davis complex $\mathrm{Flag}(W_\Gamma\mathcal{S}^f)$. Where we only connected vertices gW_T and $gW_{\tilde{T}}$ if $|T| + 1 = |\tilde{T}|$. This has been done as an example in Figure 4.3.1 for $\Gamma :=$ 

. The cubes are then of the form

$$[gW_{T_1}, gW_{T_2}] := \{gW_{\tilde{T}} \in W_\Gamma\mathcal{S}^f \mid T_1 \subseteq \tilde{T} \subseteq T_2\},$$

this would be a $\big|[gW_{T_1}, gW_{T_2}]\big|$-dimensional cube.

Remark 2.11.5. As already noticed in Example 2.8.3 we can find the fundamental domain as a part of the Davis complex C_Γ (the Davis complex in context of Definition 2.8.2). It is also clear that for the action of W_Γ on C_Γ the orbit of the fundamental domain is the whole Davis complex. The same will actually be true for the Deligne complex $A_\Gamma \mathcal{S}^f$ but now with the action of A_Γ (again by left multiplication). The following example will make this more clear.

Example 2.11.6. In Figure 2.11.1 a portion of the Davis complex of type $\Gamma :=$ $\overset{c}{\bullet} \quad \overset{a}{\bullet}\overset{2}{\text{---}}\overset{b}{\bullet}$ is drawn (as in Construction 2.11.4). In red the fundamental domain is indicated (also see Example 2.6.3 and Figure 2.6.1).

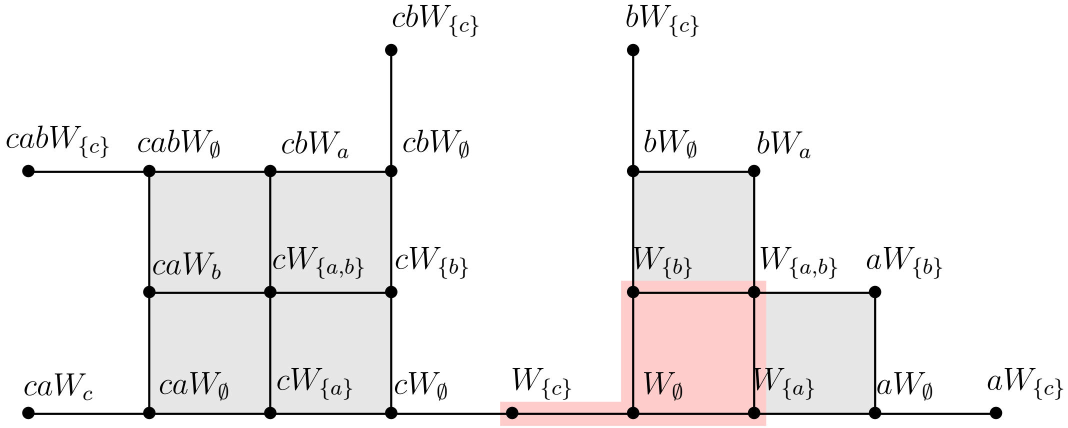


Figure 2.11.1: Cubical structure of Davis complex C_Γ.

The orbit (by left multiplication) of K by the action of W_Γ is the whole Davis complex. To construct the geometric realization of Geom $\big(\text{flag}(W_\Gamma \mathcal{S}^f)\big)$ or equivalent the barycentric subdivision of C_Γ (of the complex draw in Figure 2.8.2). We need to draw edges between the 0 cosets and 2 cosets that are included in each other (for example between $W_\emptyset$ and $W_{\{a,b\}}$) in Figure 2.11.1.

Definition 2.11.7 (Moussong metric for the Deligne complex)**.** We define a metric on the Deligne complex completely analogous to how we did for the Davis complex in Remark 2.8.9. The fundamental domain K_Γ is isomorphic to $\{C \in A_\Gamma S^f \mid \mathbb{1} \in C\}$ by

$$\mathcal{S}^f \to \{C \in A_\Gamma \mathcal{S}^f \mid \mathbb{1} \in C\} : T \mapsto A_T.$$

There is an induced metric on K_Γ in $\mathbb{R}^n$, we copy this metric to $\{C \in A_\Gamma S^f \mid \mathbb{1} \in C\}$ and extend this metric to the whole Deligne complex since $D_\Gamma = \{C \in A_\Gamma S^f \mid \mathbb{1} \in C\}^{A_\Gamma}$ (the action by left multiplication).

To end this section we refer to [40, Theorem 3.1], to see a concrete application of the Deligne complex. Herein they prove that for FC-type Artin groups the intersection of parabolic subgroups[8] is again a parabolic subgroup they use the Deligne complex extensively, while it is a priori not clear why it is needed.

2.12 Some connection between complexes and fundamental groups

In this section we will use the complexes we have discussed throughout this chapter to construct fundamental groups and quotient groups that are isomorphic to the Coxeter or Artin group.

Theorem 2.12.1. *Let W_Γ be a Coxeter group and A_Γ be the associated Artin group, let $\mathcal{B}$ be the wedge of $|V(\Gamma)|$ circles. We have the following isomorphic groups:*

(i) $W_\Gamma \cong \pi_1(\mathcal{B})/\pi_1\left(F\mathcal{S}_\Gamma^{(1)}\right) \cong \pi_1(\mathcal{S}_\Gamma)/\pi_1(F\mathcal{S}_\Gamma)$;

(ii) $A_\Gamma \cong \pi_1(\mathcal{B})/\pi_1\left(Cay(A_\Gamma)\right) \cong \pi_1(\mathcal{S}_\Gamma) \cong \pi_1(F\mathcal{S}_\Gamma/W_\Gamma)$.

Here $F\mathcal{S}_\Gamma^{(1)}$ and $Cay(A_\Gamma)$ are covering spaces of $\mathcal{B}$, and $F\mathcal{S}_\Gamma$ of $\mathcal{S}_\Gamma$. Hence, we identify these groups by the image of the natural injections $\pi_1(cay(A_\Gamma)), \pi_1\left(F\mathcal{S}_\Gamma^{(1)}\right) \hookrightarrow \pi_1(\mathcal{B})$ and $\pi_1(F\mathcal{S}_\Gamma) \hookrightarrow \pi_1(\mathcal{S}_\Gamma)$ that are induced by these covering maps.

Before we can prove this we refer to the following construction:

Theorem 2.12.2 ([48, Section 2.2.2])**.** *Let $\pi_1(\mathcal{B})$ be a fundamental group of a bouquet of circles $\mathcal{B}$. If G is a subgroup of $\pi_1(\mathcal{B})$; then G is isomorphic to a fundamental group $\pi_1(\Gamma)$ of a graph Γ such that the graph Γ covers $\mathcal{B}$. Moreover, there is an explicit construction of Γ being*

$$\begin{aligned} V(\Gamma) &:= \{fG \mid f \in \pi_1(\mathcal{B})\}; \\ E(\Gamma) &:= \{e \mid e = (fG, sfG) \text{ for } s \in E(\mathcal{B}), fG \in V(\Gamma)\}. \end{aligned}$$

In Theorem 2.12.2 an edge was a couple $(\cdot,\cdot)$ rather than a set $\{\cdot,\cdot\}$. Even though we do not care about the orientation of edges in the fundamental group of a graph, doing this with couples is important since it allows for two edges between the same pair of two vertices (since, $(a, b) \neq (b, a)$).

Proof of Theorem 2.12.1. **(i) part 1:** Since W_Γ is a group with $|V(\Gamma)|$ generators and $\pi_1(\mathcal{B})$ is the free group of rank $|V(\Gamma)|$, there is an epimorphism

$$\phi_{W_\Gamma} : \pi_1(\mathcal{B}) \twoheadrightarrow W_\Gamma.$$

Let $\ker(\phi_{W_\Gamma})$ be the kernel of this morphism. The 1-skeleton of the Davis complex is the Cayleygraph of W_Γ (by Theorem 2.8.4 (i)). We made $F\mathcal{S}_\Gamma$ by attaching to every spherical Coxeter cell oriented cells (as many as there were possible

[8] i.e. a conjugate of a special subgroup

orientations). Since an edge has only two orientations, the graph $F\mathcal{S}_\Gamma^{(1)}$ is the Cayleygraph of W_Γ where we replace every edge by two edges. However, this is exactly the graph you would become by Theorem 2.12.2.

$$\begin{aligned} V(\Gamma) :=& \{f\ker(\phi_{W_\Gamma}) \mid f \in \pi_1(\mathcal{B})\} = W_\Gamma; \\ E(\Gamma) :=& \Big\{e \mid e = \big(f\ker(\phi_{W_\Gamma}), sf\ker(\phi_{W_\Gamma})\big) \text{ for } s \in E(\mathcal{B}), f\ker(\phi_{W_\Gamma}) \in V(\Gamma)\Big\} \\ =& \big\{(w, sw), (sw, ssw) \mid s \in V(\Gamma), w \in W_\Gamma\big\} \\ =& \big\{(w, sw), (sw, w) \mid s \in V(\Gamma), w \in W_\Gamma\big\}. \end{aligned}$$

(i) part 2: Since $A_\Gamma \cong \pi_1(\mathcal{S}_\Gamma)$ has the same generator set $\{s_i \mid i \in V(\Gamma)\}$ as W_Γ with just fewer relations, there is an epimorphism

$$\begin{aligned} \psi : \pi_1(\mathcal{S}_\Gamma) \twoheadrightarrow W_\Gamma : \\ s_i \mapsto s_i \\ s_i^2 \mapsto \mathbb{1}. \end{aligned}$$

Clearly $F\mathcal{S}_\Gamma$ is a cover of the Salvetti complex, since by definition of the Salvetti complex we have

$$\chi : F\mathcal{S}_\Gamma \twoheadrightarrow F\mathcal{S}_\Gamma / W_\Gamma =: \mathcal{S}_g.$$

We choose a basepoint $x_0 \in F\mathcal{S}_\Gamma \cap \chi^{-1}(\bullet)$ (with $\bullet$ the base point of the Salvetti complex see also Definition 2.9.15 (i)). The χ map induces a monomorphism.

$$\chi^* : \pi_1(F\mathcal{S}_\Gamma, x_0) \hookrightarrow \pi_1(\mathcal{S}_g, \bullet).$$

We will now prove that $\mathrm{im}(\chi^*) = \ker(\psi)$.

"$\subseteq$": Suppose we have a closed path p in $F\mathcal{S}_\Gamma$. This path is contained in a sequence of (maximal[9]) oriented Coxeter cell $(C_1', C_2', \dots, C_k')$ (we denote C_i for the not oriented Coxeter cell in the Davis complex). The closed path p starts in a cell C_1' and ends in C_k'. Since a Coxeter cell is just a convex polytope, we can continuously deform p to a sequence of edges in C_i', see Figure 2.12.1 as example.

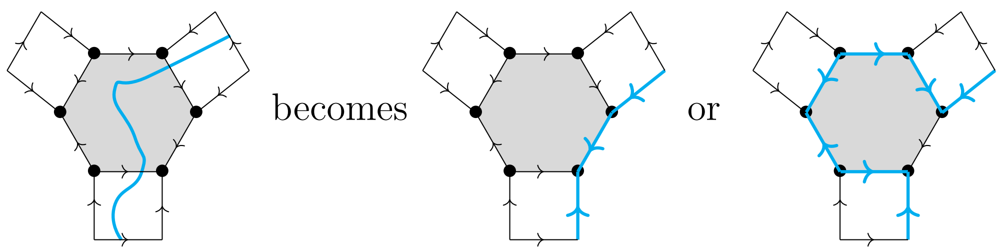


Figure 2.12.1: Continuous deformation of p.

Since oriented Coxeter cells are attached to each other along there faces, the staring and ending vertices of $\tilde{p}$ can be chosen arbitrary in the face where p

[9]i.e. there does not exist a Coxeter cell C_Δ in the Davis complex C_Γ and an index i such that C_i is a real subcomplex of C_Δ

started and ended (as long as we stay consisted in the choice in the next Coxeter cell). Hence, without loss of generality our path is a sequence of oriented edges in oriented Coxeter cells $p = (e_1, e_2, \ldots, e_m)$. Every edge is a lift of an edge in $s_i \in \pi_1(\mathcal{S}) \equiv A_\Gamma = \langle s_i \in V(\Gamma) \rangle$. Suppose this path p has a minimal amount of edges such that $\chi^*(p) \notin \ker(\psi)$. The full Salvetti complex is the Davis complex where we replace the Coxeter cells with oriented ones. Hence, there is a natural projection map

$$\pi : F\mathcal{S}_\Gamma \twoheadrightarrow C_\Gamma,$$

such that $\pi(C_i') = C_i$. One needs to be careful it could be that $C_i = C_{i+1}$ while $C_i' \neq C_{i+1}'$ (this would happen if they correspond to the same Coxeter cell but a different orientation). Let $C_1, C_2, \ldots, C_l := \pi(C_1', C_2', \ldots, C_k')$ the sequence of Coxeter cells in the Davis complex that contains $\pi(p)$ and such that $C_i \neq C_{i+1}$.

Since the Davis complex is simply connected (Remark 2.9.18 (iii)) If we would project this path $p = (e_1, e_2, \ldots, e_m)$ on the Davis complex, there would be a furthest Coxeter C_{i_0} cell from where our path goes back in the direction of the base point, i.e. $C_{i_0-1} = C_{i_0+1}$. Hence, our situation is the following picture (if it would be the case that the Coxeter cells are of dimension 2).

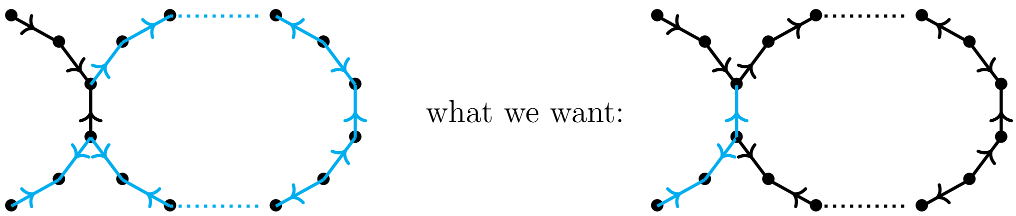


Figure 2.12.2: Furthest Coxeter cell.

Suppose C_{i_0} is the furthest Coxeter cell in where p is contained. Suppose C_Λ with that $\Lambda \subseteq \Gamma$ and W_Γ spherical is the smallest Coxeter cell in C_Γ in which $C_{i_0} \cap p$ is contained. We have w.l.o.g. $p = (e_1, e_2, \ldots, e_i, e_{i+1}, \ldots e_{i+l}, \ldots, e_m)$ where $e_i, e_{i+1}, \ldots, e_{i+l} \in C_\Lambda$.

$$\begin{aligned}
\phi\left(\chi^*(p)\right) &= \phi\left(\chi(e_1, e_2, \ldots e_m)\right) \\
&= \psi(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_i^{\epsilon_i} \ldots s_{i+l}^{\epsilon_{i+k}} \cdots s_m^{\epsilon_m}) \quad \text{where } \epsilon_i \in \{-1, +1\} \\
&= s_1 s_2 \cdots s_i \cdots s_{i+l} \cdots s_m \\
&= s_1 s_2 \cdots (s_i \cdots s_{i+l})^{-1} \cdots s_m \\
&= s_1 s_2 \cdots (s_i' \cdots s_{i+h}') \cdots s_m \\
&= \psi(\chi^*(p')),
\end{aligned}$$

where $(s_i \cdots s_{i+l})^{-1} s_i' \cdots s_{i+h}' = \mathbb{1}$ in W_λ. Since the path which would correspond to $s_i' \cdots s_{i+h}'$ in the Coxeter cell C_Λ would be in one of its faces (in Figure 2.12.2 it contains only one blue edge since there our Coxeter cell only contained one edge), it is strictly smaller in length then $s_i \cdots s_{i+l}$, hence $h < l$. Hence, we proved that

the image of p is the same as the image of this path shortened by replacing the path in the furthest Coxeter cell. Since p was minimal, then $\chi^*(p') \in \mathrm{im}(\chi^*)$ implies $(\chi^*(p') \in \ker(\psi)$ and hence,$(\chi^*(p) \in \ker(\psi)$ which is a contradiction.
"$\supseteq$": Suppose we have a path p in $\pi_1(\mathcal{S}_\Gamma)$ such that $\psi(p) = \mathbb{1} \in W_\Gamma$. Hence, $p = (s_1^{\epsilon_1}, e_2^{\epsilon_2}, \ldots, e_m^{\epsilon_m})$ such that

$$\begin{aligned}\psi(p) &= \psi(s_1^{\epsilon_1}, e_2^{\epsilon_2}, \ldots, e_m^{\epsilon_m}) \\ &= s_1 s_2 \cdots s_m = \mathbb{1} \quad \text{in } W_\Gamma.\end{aligned}$$

This means that if we track down the path in the Davis complex / Cayleygraph of W_Γ. This path starts at $\mathbb{1}$ and ends at $\mathbb{1}$, since the whole element $s_1 s_2 \cdots s_m$ is $\mathbb{1}$. This closed path we copy it to $F\mathcal{S}_f$ where we rescue the orientation by ϵ the obtained path we call $\bar{p}$. It clearly follows that the image under χ of the path $\bar{p}$ is p. Hence, we conclude $W_\Gamma \cong \pi_1(\mathcal{S}_\Gamma)/\ker(\psi) = \pi_1(\mathcal{S}_\Gamma)/\mathrm{im}(\chi^*) = \pi_1(\mathcal{S}_\Gamma)/\pi_1(F\mathcal{S}_\Gamma)$.
(ii): By Theorem 2.9.17 we already know that $A_\Gamma \cong \pi_1(\mathcal{S}_\Gamma) \cong \pi_1(F\mathcal{S}_\Gamma/W_\Gamma)$. We prove that $A_\Gamma \cong \pi_1(\mathcal{B})/\pi_1\big(Cay(A_\Gamma)\big)$. We go analog like previous parts, we prove that the kernel of the following morphism is $\pi_1\big(Cay(A_\Gamma)\big)$

$$\phi_{A_\Gamma} : \pi_1(\mathcal{B}) \twoheadrightarrow A_\Gamma.$$

The kernel is by Theorem 2.12.2 the fundamental group of the graph

$$\begin{aligned}V(\Gamma) &:= \big\{f\ker(\psi_{A_\Gamma}) \mid f \in \pi_1(\mathcal{B})\big\} = A_\Gamma; \\ E(\Gamma) &:= \big\{e \mid e = \{f\ker(\phi_{A_\Gamma}), sf\ker(\psi_{A_\Gamma})\} \text{ for } s \in E(\mathcal{B}), f\ker(\psi_{A_\Gamma}) \in V(\Gamma)\big\} \\ &= \big\{\{a, sa\} \mid s \in V(\Gamma), a \in A_\Gamma\big\}.\end{aligned}$$

This graph is exactly the Cayleygraph. □

Definition 2.12.3 ([42]). Consider the natural epimorphism (that maps generators to generators) $\phi : A_\Gamma \twoheadrightarrow W_\Gamma$. The kernel of this morphism is the *colored Artin group*, which is hence also the fundamental group $\pi_1(F\mathcal{S}_\Gamma)$.

3 The $K(\pi, 1)$ conjecture

In this section we will explain the $K(\pi, 1)$ conjecture. We will not go into detail, and won't do much more than state this conjecture. Instead, we will convince you from the fact that our complexes discussed in Chapter 2 will be useful, and will give us some equivalent statements of the $K(\pi, 1)$ conjecture.

3.1 Definitions

Definition 3.1.1. Let G be a group and X a topological space such that for certain $k \in \mathbb{N}$ we have homotopy groups $\pi_k(X) \cong G$ and $\pi_n(X) = \{\mathbb{1}\}$ for all $n \in \mathbb{N} \setminus \{k\}$. Then we call X a $K(G, k)$ space. In the case that $k = 1$ we call X an *Eilenberg-MacLane Space.*

Definition 3.1.2. A topological space X is *aspherical* if every homotopy group $\pi_k(X)$ is trivial for $k \geq 2$.

Lemma 3.1.3 ([2, Chapter 11])**.** *Suppose X a topological space and G a group. Then the following are equivalent:*

(i) X is an $K(G, 1)$ space;

(ii) X is aspherical and $\pi_1(X) = G$;

(iii) the universal cover of X is contactable and $\pi_1(X) = G$.

Theorem 3.1.4 (Cartan-Hadamard Theorem [14, Theorem I.6])**.** *Let (X, d) be a complete connected locally* CAT(0) *metric space, then there is a unique metric $\tilde{d}$ on the universal cover $\bar{X}$ of X such that the following holds:*

(i) *the covering map $\bar{X} \to X$ is a local isometry;*

(ii) *$(\bar{X}, \tilde{d})$ is a* CAT(0) *space.*

The metric $\tilde{d}$ coincides with the metric induced[10] *by d on $\bar{X}$.*

[10] This metric is precisely the piecewise metric you get from lifting paths from X to $\bar{X}$, see

3.2 A $K(\pi, 1)$ space in the finite case

Construction 3.2.1. Suppose W_Γ is a finite Coxeter group of type Γ with $n := |V(\Gamma)|$. By Definition 1.2.4, there is an action of W_Γ on $\mathbb{R}^n$. For every $s \in V(\Gamma)$ there is a unique fix hyperplane $H_s \subseteq \mathbb{R}^n$ (the fix hyperplane of the map r_s, see Definition 1.2.4). Denote $\mathbb{C}H_s$ for the subspace $H_s \otimes_{\mathbb{R}} iH_s$. Using $\mathbb{R}^n \subset \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$, one can define a natural action of W_Γ on $\mathbb{C}^n$. Consider the space

$$\mathcal{Z}_\Gamma := \mathbb{C}^n \setminus \bigcup_s \mathbb{C}H_s.$$

Then clearly W_Γ acts freely on this space.

From now one we denote $\mathcal{H}_\Gamma := \cup_s \mathbb{C}H_s$ to be the set of fix hyperplanes.

Example 3.2.2. Consider W_Γ, the Coxeter group of type[11] $\mathcal{A}_{n-1}$. This means

$$W_\Gamma = \left\langle s_1, s_2, \dots, s_{n-1} \mid s_i^2 = \mathbb{1}, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \underbrace{s_i s_j = s_j s_i}_{\forall i,j,|i-j|>2} \right\rangle \cong \mathsf{Sym}_n.$$

Using Construction 3.2.1 we want to determine the pace $\mathcal{Z}_\Gamma / W_\Gamma$. Because Sym_n is generated by transpositions $(i, j) \in \mathsf{Sym}_n$, we identify these transpositions with reflections in $\mathbb{C}^n$ sending $(z_1, z_2, \dots, z_i, \dots, z_j, \dots, z_n)$ to $(z_1, z_2, \dots, z_j, \dots, z_i, \dots, z_n)$ (one transposition for every pair (i, j) with $1 \le i < j \le n$). Vectors in $\mathbb{C}^n$ for which two coordinate values are equal (say $z_i = z_j$) are hence not contained inside $\mathcal{Z}_\Gamma$. This is the case because they are contained in a fix hyperplane corresponding to the reflection $(i, j) \in \mathsf{Sym}_n$ (this is the reflection that sends the two coordinates to each other). Thus, $\mathcal{Z}/W_\Gamma$ is just the space of subsets of size n of $\mathbb{C}$ (i.e. $\mathcal{Z}_\Gamma / W_\Gamma = \big\{ \{z_1, z_2, \dots, z_n\} \mid z_i \neq z_j \in \mathbb{C} \big\}$). The fundamental group of $\mathcal{Z}_\Gamma / W_\Gamma$ can be seen as follows: a closed path in $\mathcal{Z}_\Gamma / W_\Gamma$ is a set of n closed paths (or "strands") in $\mathbb{C}$, one for every coordinate, with the condition that they never intersect (two coordinates are never the same) at the end we go back at our starting element. In Figure 3.2.1 is a possible closed path drawn, that consists of n disjoint closed paths (you start at the top and as you walk along the path in $\mathcal{H}_\Gamma / W_\Gamma$ at every time there are n new different elements of $\mathbb{C}$). The resulting fundamental group is the n-strand braid group that has presentation:

$$\pi_1(\mathcal{Z}_\Gamma / W_\Gamma) = \left\langle s_1, s_2, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \underbrace{s_i s_j = s_j s_i}_{\forall i,j,\ |i-j|>2} \right\rangle \cong A_\Gamma. \quad (3.1)$$

So by this construction we get back our Artin group.

[12, Definition 3.24].

[11] Here we use the well-know naming of the finite Coxeter groups which can be found in a lot of literature, or on Wikipedia.

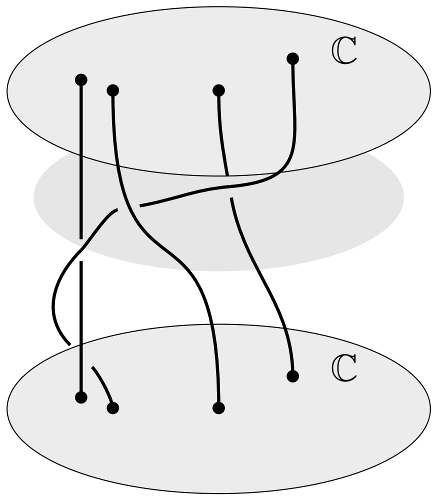


Figure 3.2.1: A path in the fundamental group of $\mathcal{Z}_\Gamma/W_\Gamma$.

In Example 3.2.2 we see that equation (3.1) can be uses to construct a space such that $\pi_1(\mathcal{Z}_\Gamma/W_\Gamma) = A_\Gamma$ when Γ is of type $\mathcal{A}_{n-1}$. One can do this in general for finite type Artin groups.

Theorem 3.2.3 ([21, page 1])**.** *Suppose W_Γ a finite Coxeter group and let $\mathcal{Z}_\Gamma$ be the space constructed in Construction 3.2.1. The space $\mathcal{Z}_\Gamma/W_\Gamma$ is a $K(A_\Gamma, 1)$ space.*

Proof. First Brieskorn proved that the fundamental group of $\mathcal{Z}_\Gamma/W_\Gamma$ is A_Γ in [13]. Deligne proved the stronger result in [25, (4.4)Théoréme]. □

3.3 A $K(\pi, 1)$ space in the infinite case

For the infinite case we need to restrict Construction 3.2.1 to a cone in $\mathbb{R}^n$. One of the reasons our previous construction does not work anymore is that there could be points in $\mathbb{R}_n$ that have an infinite stabilizer from the action of W_Γ (by its Tits representation).

Definition 3.3.1 ([21, page 598])**.** Let cone(Γ) be the simplicial cone in $\mathbb{R}^n$ from Definition 1.2.10. The *Tits cone* Tits(Γ) is the following space

$$\text{Tits}(\Gamma) := \bigcup_{w \in W_\Gamma} \text{cone}(\Gamma)^w.$$

In the finite case we have $\text{Tits}(\Gamma) = \mathbb{R}^n$. It happens to be the case that

$$\text{int}\big(\text{Tits}(\Gamma)\big) \cap \text{cone}(\Gamma) = \{x \in \text{cone}(\Gamma) \mid W_x \text{ is finite}\}.$$

Definition 3.3.2 (complex Tits cone [39, Section 4])**.** Let W_Γ be a Coxeter group let $\mathcal{H}$ the set of fix hyperplanes (of the linear maps r_i in Definition 1.2.4). Consider $\mathbb{R}^n \subset \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$, the *complexified Tits cone* is $\mathbb{C}\mathrm{Tits}(\Gamma) := \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Tits}(\Gamma)$. Clearly W_Γ acts freely on $\mathrm{int}\big(\mathbb{C}\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H} =: \mathcal{Z}_\Gamma$

Theorem 3.3.3 ([39, Theorem 4.7]). *The interior of the complexified Tits cone with fix hyperplanes removed* $\mathbb{C}\,\mathrm{int}\big(\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H}$ *is equivariantly homotopy equivalent to the full Salvetti complex* $F\mathcal{S}_\Gamma$*, for the action of* W_Γ.

We have the following summary.

Remark 3.3.4 ([39, page 18]). For a general Coxeter group W_Γ, with C_Γ its Davis complex and $F\mathcal{S}_g$ the full Salvetti complex, we have

(i) $\mathrm{int}\big(\mathrm{Tits}(\Gamma)\big) \cong C_\Gamma$;

(ii) $\mathbb{C}\mathrm{int}\big(\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H} \cong F\mathcal{S}_\Gamma$.

The following conjecture can be found in [17, Conjecture 1]

Conjecture 3.3.5 (The $K(\pi,1)$ conjecture). *The* $K(\pi,1)$ *conjecture states that one of the following equivalent assertions is true for any Coxeter group* W_Γ*:*

(i) *the space* $\Big(\mathbb{C}\,\mathrm{int}\big(\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H}\Big)/W_\Gamma$ *is a* $K(A,1)$*-space;*

(ii) *the space* $F\mathcal{S}_\Gamma$ *is a classifying space for* A_Γ*;*

(iii) *the universal cover of* $F\mathcal{S}_\Gamma$ *(or equivalent of* $\mathcal{S}_\Gamma$*) is contactable;*

(iv) *the universal cover of* $\mathrm{int}\big(\mathbb{C}\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H}$ *(or equivalent of* $\Big(\mathrm{int}\big(\mathbb{C}\mathrm{Tits}(\Gamma)\big) \setminus \mathcal{H}\Big)/W_\Gamma$*) is contractible.*

(v) [21, page 3 or 599] *the Deligne complex* $\mathrm{Geom}\big(\mathrm{Flag}(A_\Gamma \mathcal{S}^f)\big)$ *being contractible (for the Moussong metric Definition 2.11.7).*

Part of the $K(\pi,1)$ conjecture is already proven by van der Lek.

Theorem 3.3.6. *For any Coxeter group* W_Γ *(also the infinite once) with associated Artin group* A_Γ*, the fundamental group of* $\mathcal{Z}_\Gamma/W_\Gamma$ *is isomorphic to* A_Γ*.*

Proof. See [49]. □

We will now discuss some classes of (infinite) Artin groups where the $K(\pi,1)$-conjecture already proved

Definition 3.3.7 ([38, Definition 2.3]). An Artin group A_Γ is two-dimensional if for every triangle $\Delta \subseteq \Gamma$ in the defining graph, the group A_Δ is infinite.

The following theorem was proven by Charney and Davis.

Theorem 3.3.8 ([28, Theorem 2.7], [21, Theorem B]). *The* $K(\pi,1)$ *conjecture holds for every two dimensional Artin group.*

Another class of Artin groups for which the $K(\pi,1)$ conjecture is solved is the following.

Theorem 3.3.9 ([21, Theorem A]). *Let A_Γ be an Artin group of FC-type, then the $K(\pi, 1)$ conjecture holds.*

Skech of proof. [20, page 6 & 8]: In Construction 2.11.4 we gave a cubical structure to the Deligne complex. We can then prove that in the FC-types Gromov's Link Considtion (Theorem 2.3.10) is satisfied. Then by Theorem 2.3.7, the Deligne complex is contractible and by the equivalences in Conjecture 3.3.5 we are done. □

Since every right-angled Artin group is of FC-type, the $K(\pi, 1)$ conjecture holds for right-angled Artin groups. However, there is another way one can see that the $K(\pi, 1)$ conjecture holds for these groups.

Theorem 3.3.10. *Let A_Γ be a right-angled Artin group, then the $K(\pi, 1)$ conjecture holds.*

Sketches of proofs. We give multiple proofs:
(1): Follows directly from Theorem 3.3.9.
(2): Let A_Γ be a right-angled Artin group. The Salvetti complex is in this case a cube complex (see later 5.2.6). Then by Theorem 2.3.9 is $\mathcal{S}_\Gamma$ a locally CAT(0) space. By Theorem 3.1.4 the universal cover $\bar{\mathcal{S}}_\Gamma$ is CAT(0). Finely by Theorem 2.3.7 the universal cover is contractible, which is what we wanted to prove.
(3): In Chapter 4.5 we will prove that the Deligne complex of a right-angled Artin group is the geometric realization of a building. It then will follow from Theorem 4.2.9 that the Deligne complex is a complete CAT(0) space, the result then follows from Theorem 2.3.7. One important detail here is that the Moussong metric on the Deligne complex (Definition 2.11.7) and the metric on the building coincide (see Definition 4.2.8). □

For general Artin groups, some simple question remains unsolved.

Conjecture 3.3.11. *Let A_Γ be an arbitrary Artin group, then the following are satisfied*

(i) *A_Γ is torsion-free;*

(ii) *A_Γ has solvable word problem;*

(iii) *A_Γ is linear;*

(iv) *if A_Γ is irreducible (i.e. $A_\Gamma \neq A_{\Gamma_1} \times A_{\Gamma_2}$), then we have*

$$Z(A_\Gamma) = \begin{cases} \mathbb{Z} & \text{if } A_\Gamma \text{ is of finite type;} \\ \{\mathbb{1}\} & \text{if } A_\Gamma \text{ is of infinite type.} \end{cases}$$

.

All the conjectures in Conjecture 3.3.11 have a positive answer if A_Γ is of finite type. Part (ii) is solved for Artin groups of FC-type [3]. For some more results we refer to [11].

4 Buildings

In the theory of Artin groups, if the Artin group is right-angled; then we will encounter buildings. Since the geometric realization of a building will be CAT(0), this section will also give a proof for Theorem 3.3.10. Moreover, being CAT(0) will make sure the constructions in Chapter 5 will work, and this will then be crucial in some quasi-isometric properties of right-angled Artin groups in Chapter 6.

4.1 Definitions

There are multiple equivalent definitions of a building (or Tits building). In this thesis, we will use the one from the book by M. Davis [24]. First we define the set of reduced words where the alphabet is the set of generators of a Coxeter group W_Γ.

Definition 4.1.1. Let W_Γ be a Coxeter group. Let $S := V(\Gamma)$.

1. Let S^* be the set of all possible finite words in S, i.e.,

$$S^* := \{s_1 s_2 \cdots s_n \mid n \in \mathbb{N} \ \& \ s_i \in S\};$$

2. [24, Definition 3.4.1] An *elementary M-operation* on a word in S^* is one of the following operations: either deleting a subword of the form ss or replacing a subword of the form $s\tilde{s}s\tilde{s}\cdots$ of length $m_{s,\tilde{s}}$ to $\tilde{s}s\tilde{s}s\cdots$ of length $m_{s\tilde{s}}$.
3. A word is *M-reduced* if it cannot[12] be shortened in length by some M-operations.

Definition 4.1.2. Consider a set S. A *chamber system over the set S* is a pair $(C, S) =: \Phi$ with C a set such that for every $s \in S$ there is an equivalence relation $R_s \subseteq C \times C$ on C. We call the elements of C *chambers*.

[12] If $m_{s\tilde{s}} = 2$ the words $s\tilde{s}$ and $\tilde{s}s$ are two different M-reduced words.

(i) Two chambers c and c' are *s-adjacent* if they are *s-equivalent*. In this case, we also denote $c \equiv_s c'$.

(ii) If for c and $c' \in C$ there exists $c_0, c_1, \ldots, c_n$ such that $c = c_0 \equiv_{s_1} c_1 \equiv_{s_2} \cdots \equiv_{s_n} c_n = c'$ for $\bar{s} := s_1 s_2 \cdots s_n \in S^*$, then we call $(c_0, c_1, \ldots, c_n)$ a *$\bar{s}$-gallery* connecting c with c'.

(iii) For a subset $T \subseteq S$, consider $\mathcal{R}_T \subseteq C$ a subset of chambers such that $c \in \mathcal{R}_T$ if and only if $\forall c' \in \mathcal{R}_T$, there is a $T^* \ni \bar{t}$ gallery connecting c with c'. Then $\mathcal{R}_T$ is called a *T-residue* (or *T-connected component*). If T is a singleton $T = \{s\}$; then $\mathcal{R}_s$ is called a *s-panel*.

(iv) For $T \subseteq S$ and $c \in C$, we denote $[c]_T$ for the T-connected component containing c.

(v) The *rank* of a *T-residue* is $|T|$.

(vi) A residue $\mathcal{R}_T$ is *spherical* if W_T is finite.

Example 4.1.3. Consider a group G with a family of subgroups $\forall s \in S, H_s \leq G$, and a subgroup B such that $B \leq H_s \leq G$. Then we define a chamber system $\Phi(G, B, \{H_s\}_{s \in S})$ as follows: The set of chambers is the set of cosets in G/B. Two chambers gB and $g'B$ are s-adjacent if and only if $gH_s = g'H_s$.

Definition 4.1.4. Suppose W_Γ is a Coxeter group. A chamber system $\Phi = (C, S)$ over $S = V(\Gamma)$ is a *building of type* Γ if it satisfies the following conditions:

(i) Every s-panel contains at least 2 chambers, i.e. $\forall s \in S, \forall c_1 \in C, |[c]_s| \geq 2$;

(ii) There exists a map $\delta : C \times C \to W$ called the *W_Γ-valued distance function*, satisfying

$$(\forall c_1, c_2 \in C)(\forall \bar{s} \in S^* \text{ M-reduced})\Big(w(\bar{s}) = \delta(c_1, c_2) \\ \Leftrightarrow \text{there is a } \bar{s} \text{ gallery connecting } c_1 \text{ with } c_2\Big),$$

where $w(\bar{s}) := s_1 s_2 \cdots s_n \in W_\Gamma$ as value in W_Γ.

If a chamber system is a building we will use the notation $\mathcal{B}$ instead of Φ.

You can check yourself that the chamber system in the following definition is a building.

Definition 4.1.5 (Abstract Coxeter complex)**.** For W_Γ a Coxeter group, the chamber system $\mathcal{B}_{W_\Gamma} := \Phi\left(W_\Gamma, \{\mathbb{1}\}, (W_{\{v\}})_{v \in V(\Gamma)}\right)$ is called the *abstract Coxeter complex* of W_Γ. This is a building of type Γ for the following distance function

$$\delta_{W_\Gamma} : (w_1, w_2) \mapsto w_1^{-1} w_2.$$

This building is important, since it will always be the smallest possible building of a given type. Moreover, it only consists of one apartment (see Definition 4.1.7).

Definition 4.1.6. Let $\mathcal{B}$ be a building of type Γ.

(i) A residue $\mathcal{R}_T$ (with $T \subseteq S = V(\Gamma)$) is a *spherical residue* if W_T is finite.

(ii) The *chamber graph* $\Lambda_{\mathcal{B}}$ of $\mathcal{B}$ is the graph with vertices $V(\Lambda_{\mathcal{B}}) := C$ (the chambers of $\mathcal{B}$). There is a labeled edge between c and c' with label $s \in V(\Gamma)$ if and only if they are s-adjacent.

Definition 4.1.7 (Apartment)**.**] Consider $\mathcal{B} = (C, S)$ a building of type Γ. An *apartment* in (C, S) is the image of a map $\alpha : W_\Gamma \to C$ which satisfies the following:

$$(\forall w_1, w_2 \in W_\Gamma) \left(\delta_{\mathcal{B}}\left(\alpha(w_1), \alpha(w_2')\right) = \delta_{W_\Gamma}(w_1, w_2)\right).$$

Property 4.1.8 ([44, (3.7) Corollary])**.** *In a building any two chamber are contained in a common apartment.*

4.2 Geometric realization

Definition 4.2.1. Let $\mathcal{B}$ be a building of type Γ and let $\mathcal{C}(\mathcal{B}) := \{\mathcal{R}_T \mid T \subseteq V(\Gamma), W_T \text{ is spherical}\}$ be the set of spherical residues now seen as a poset (by inclusion). The *geometric realization* of $\mathcal{B}$ is

$$\mathrm{Geom}(\mathcal{B}) := \mathrm{Geom}\left(\mathrm{Flag}\left(\mathcal{C}(\mathcal{B})\right)\right).$$

This geometric realization is also called the Davis realization.

Remark 4.2.2. The abstract Coxeter complex $\mathcal{B}_{W_\Gamma}$ from Definition 4.1.5 is the smallest building of type Γ. This building only consists of one apartment. The connected components or residues are of the form gW_Λ with $\Lambda \subseteq \Gamma$, so the spherical residues correspond to the spherical cosets, i.e. $\mathcal{C}(\mathcal{B}_{W_\Gamma}) = W_\Gamma\mathcal{S}^f$.

Example 4.2.3. (i) The chamber graph of $\mathcal{B}_{W_\Gamma}$ of type $\Gamma :=$ $\overset{a}{\bullet}$—$\overset{b}{\bullet}$ is given in Figure 4.2.1. The vertices correspond to the elements of W_Γ, which are the chambers of $\mathcal{B}$. The label s of an edge between w and w' corresponds with elements of $V(\Gamma)$ such that $w \equiv_s w'$, $s \in V(\Gamma)$.

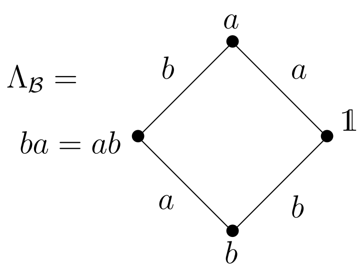


Figure 4.2.1: Chamber graph of one apartment of type Γ.

$$\begin{aligned}
\{\mathbb{1}, b\} &= \mathcal{R}_b &&= W_{\{b\}}\\
\{\mathbb{1}, a\} &= \mathcal{R}_a &&= W_{\{a\}}\\
\{a, ab\} &= \mathcal{R}'_b &&= aW_{\{b\}}\\
\{a, ab\} &= \mathcal{R}'_a &&= bW_{\{a\}}\\
\{\mathbb{1}, a, b, ab\} &= \mathcal{R}_{a,b} &&= W_{\{a,b\}}
\end{aligned}$$

Clearly the chamber graph (Figure 4.2.1) of this building coincides with the Cayleygraph of W_Γ and with the 1-skeleton of the Davis complex of W_Γ. Because we have a clear understanding of the spherical residues of this building one easily compute the geometric realization In figure 4.2.2.

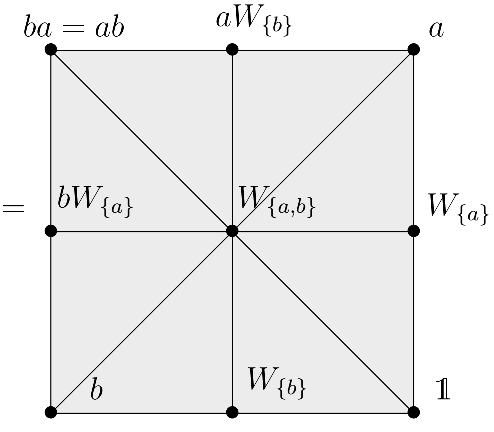
$\mathrm{Geom}(\mathcal{B}_{W_\Gamma}) =$ 

Figure 4.2.2: Geometric realization of an apartment of type Γ.

This geometric realization is exactly the same as geom $\big(\mathrm{Flag}(W_\Gamma\mathcal{S}^f)\big)$.

(ii) Suppose $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$. Then the geometric realization of one apartment is the following simplicial complex.

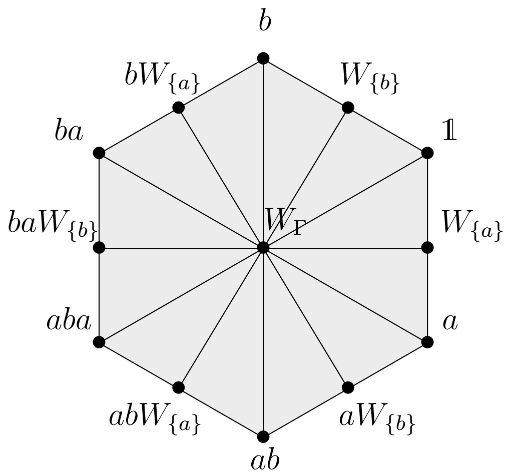
$\mathrm{Geom}(\mathcal{B}_{W_\Gamma}) =$ 

Figure 4.2.3: The geometric realization of one apartment of type Γ

This is again the same as geom $\big(\mathrm{Flag}(W_\Gamma\mathcal{S}^f)\big)$, and also as the barycentric subdivision of the Davis complex in Example 2.7.3.

Remark 4.2.4. In Construction 2.11.1 we give a different construction of the Davis complex (this was the dual to the convex polytope constructed in Definition

2.7.1). Here it does not matter what construction we use, since the barycentric subdivision of dual convex polytopes are isomorphic as simplicial complexes.

Theorem 4.2.5 ([44, (3.5) Theorem])**.** *Let $\mathcal{B}$ be a building of type Γ and $\mathcal{R}_\Lambda$ a residue of type $\Lambda \leq \Gamma$. Then we have the following*

(i) *For every two $x, y \in \mathcal{R}_\Lambda$, we have $\delta_{\mathcal{B}}(x, y) \in W_\Lambda$.*

(ii) *The residue $\mathcal{R}_\Lambda$ is a building of type Λ.*

Hence, it also makes sense to talk about $\mathrm{Geom}(\mathcal{R}_T)$ as subcomplex of $\mathrm{Geom}(\mathcal{B})$.

Lemma 4.2.6. *Let $\mathcal{B}_\Gamma$ be the building of type Γ consisting of one apartment, and let C_Γ be the Davis complex of W_Γ. Then we have that*

$$\mathrm{Geom}(\mathcal{B}_\Gamma) = \text{ barycentric subdivision of } C_\Gamma.$$

Remark 4.2.7. Here we extend the work done in Remark 2.7.6. We have that

$$\mathrm{Geom}(\mathcal{B}_{W_\Gamma}) \cong \mathrm{Geom}\Big(\mathrm{Flag}\left(\mathrm{Face}(C_{W_\Gamma})\right)\Big) \cong \mathrm{Geom}\Big(\mathrm{Flag}(W_\Gamma \mathcal{S}^f)\Big).$$

This means in some sense the Davis complex of W_Γ is a building of type Γ even more so it is the smallest building of type Γ because it only consists of one apartment. In Section 4.5 we will deduce that there exists a building $\mathcal{B}_{A_\Gamma}$ such that $\mathrm{Geom}(\mathcal{B}_{A_\Gamma}) \cong \mathrm{Geom}(\mathrm{Flag}(A_\Gamma \mathcal{S}^f))$ for a right-angled Artin group A_Γ.

It will turn our that the geometric realization of a building is CAT(0), however first we need to define what metric we will use on these structures. One possibility is the induced metric on the simplices like in Example 2.1.3, however this metric is to general and does not use the geometric structure of W_Γ.

Definition 4.2.8 (Piecewise euclidean metric on $\mathrm{Geom}(\mathcal{B})$, [24, page 337])**.** We will define the following metric on the geometric realization of a building $\mathcal{B}_\Gamma$ of type Γ. In Remark 2.8.9 we defined the metric on the Davis complex as piecewise metric induced by the orbit of K_Γ by the action of W_Γ. One can prove ([24, Section 18.2]) that there is an isomorphism between the geometric realization of a chamber c (i.e. $\mathrm{Geom}\big(\{F \in C(\mathcal{B}) \mid c \in F\}\big)$) and $K_\Gamma := \mathrm{geom}\Big(\mathrm{flag}(\mathcal{S}^f_\Gamma)\Big)$ (Definition 2.6.1). Hence, just as for the Davis complex we have that K tiles whole $\mathrm{geom}(\mathcal{B})$. We define the metric on $\mathrm{geom}(\mathcal{B})$, as the piecewise metric of K.

Theorem 4.2.9 ([24, Theorem 18.3.1])**.** *The geometric realization of any building is a complete* CAT(0) *space.*

Corollary 4.2.10 ([24, Corollary 18.3.7])**.** *The geometric realization of any building is contractible.*

Proof. This is the case for all CAT(0) spaces Theorem 2.3.7. □

Remark 4.2.11. For right-angled Coxeter groups, we can define the geometric realization of $W_\Gamma \mathcal{S}^f$ including its Moussong metric also differently using the notion of graph product (Definition 2.9.1) one can define the cubical representation of the fundamental domain as in Definition 2.6.1 and Remark 2.6.2 also as $\tilde{\prod}_{v\in V(\Gamma)}([0,1],0)$. Sometimes this is also called the *Davis chamber* because it is isomorphic to the geometric realization of a chamber i.e. the subcomplex of the apartment $\mathcal{B}_{W_\Gamma}$ spanned by the cosets $C \in W_\Gamma \mathcal{S}^f$ that contain a fixed chamber $c \in \mathcal{B}_{W_\Gamma}$. We extend this metric the like we did in Remark 2.8.9.

4.3 Right-angled buildings

In the upcoming section a building we will always be a right-angled building, unless stated otherwise. The theory of right-angled buildings will be useful in Chapter 5. For right-angled Artin groups the Deligne complex will be a right-angled building (Section 4.5).

Example 4.3.1. In Example 4.1.3 we constructed the geometric realization of one apartment of type $\Gamma := \overset{a}{\bullet}\overset{2}{\text{—}}\overset{b}{\bullet}$ in Figure 4.2.2. If we remove the edges between the rank 0 residue and the rank 2 residue, we get a cube complex shown in Figure 4.3.1.

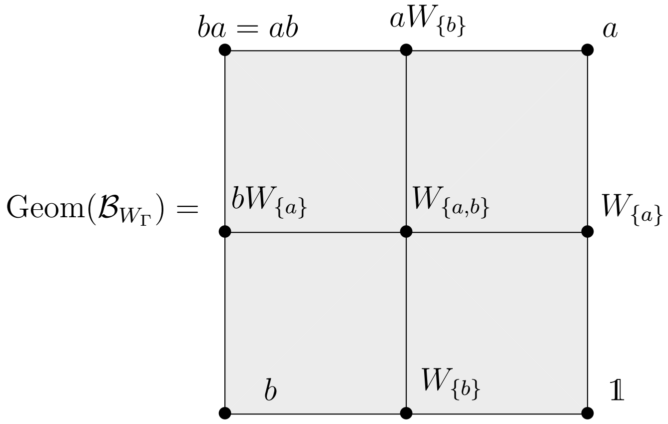


Figure 4.3.1: Geometric realization of an apartment of type Γ.

Definition 4.3.2. A building $\mathcal{B}$ of type Γ is a *Right-angled building* if for every $i \neq j \in V(\Gamma)$, $m_{ij} \in \{2, \infty\}$.

Right-angled buildings are fun and easy to work with comparing the arbitrary buildings. This is because we can define some ideas such as parallel residues. They are also easy to construct, Example 4.1.3 gives a way to construct chamber systems. One can use this to find a Right-angled building for every choice of the cardinality of the panels (in Construction 4.3.7).

Definition 4.3.3. Let $\mathcal{B}$ a building of type Γ.

(i) For two chambers c_1 and c_2, we denote $d(c_1, c_2)$ to be the minimal length of a word in w_Γ that represent $\delta(c_1, c_2)$.

(ii) We define the distance between two residues $\mathcal{R}_1$ and $\mathcal{R}_2$ to be the minimal distance between chambers of them, i.e.

$$d(\mathcal{R}_1, \mathcal{R}_2) := \inf\{d(c_1, c_2) \mid c_1 \in \mathcal{R}_1, c_2 \in \mathcal{R}_2\}$$

.

(iii) For every residue $\mathbb{R}$ there is a projection map $proj_{\mathcal{R}} : \mathcal{B} \to \mathcal{R}$, where $proj_{\mathcal{R}}(c) \in \mathcal{R}$ is the unique[13] chamber c' in $\mathcal{R}$ such that $d(c, c') = d(c, \mathcal{R})$.

(iv) Two residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel if we have $proj_{\mathcal{R}_1}(\mathcal{R}_2) = \mathcal{R}_1$ and $proj_{\mathcal{R}_2}(\mathcal{R}_1) = \mathcal{R}_1$.

Similar to how we could construct a cubical structure on the Davis and Deligne complex (Construction 2.11.4), we can do the same for right-angled buildings.

Construction 4.3.4. Let $\mathcal{B}$ be a right-angled building, define the cubes as follows

$$[\mathcal{R}_1, \mathcal{R}_2] := \{\mathcal{R} \in \mathcal{C}(\mathcal{B}) \mid \mathcal{R}_1 \subseteq \mathcal{R} \subseteq \mathcal{R}_2\},$$

this would be a $\Big(\big|[\mathcal{R}_1, \mathcal{R}_2]\big| - 1\Big)$-dimensional cube.

This cubical structure coincide to the geometric realization $\mathrm{Geom}(\mathcal{B})$ where we use in Definition 4.2.1 the cubical structure of the fundamental domain / Coxeter block from Definition 2.6.1. See also Remark 4.2.11.

Lemma 4.3.5 ([33, Lemma 3.12])**.** *If two residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel then* $\mathrm{Geom}(\mathcal{R}_1)$ *and* $\mathrm{Geom}(\mathcal{R}_2)$ *are parallel for the* $\mathrm{CAT}(0)$ *metric.*

Definition 4.3.6. A building $\mathcal{B}$ is *semi-regular* if for all $s \in S$ the cardinality of different s-panels is the same being $q_s (\geq 2)$. In this case we say that $\mathcal{B}$ has *thickness* $\vec{q} := (q_s)_{s \in S}$.

Construction 4.3.7. [15] Choose Γ a right-angled graph and $\vec{q} = (q_s)_{s \in V(\Gamma)}$ arbitrary. Let X_s be an arbitrary group of cardinality q_s. Define

$$G_\Gamma(\vec{X}) := \mathop{\ast}_{s \in S} X_s / \left\langle [X_s, X_t] \mid s, t \in S, s \sim t \text{ in } \Gamma \right\rangle.$$

Now consider $\Phi\left(G_\Gamma(\vec{X}), \{\mathbb{1}\}, (X_s)_{s \in S}\right)$ as in Example 4.1.3. We claim that this is a right-angled building where the cardinality of every s-panel is precisely $|X_s/\{\mathbb{1}\}| = |X_s| = q_s$.

Theorem 4.3.8. *The chamber system constructed in Construction 4.3.7 is a Right-angled building of type Γ and thickness $\vec{q}$.*

Proof. See [23, Theorem 4.2 & Theorem 5.1]. □

Theorem 4.3.9 ([30, Proposition 1.2])**.** *For every right-angled type Γ and thickness $\vec{q} = (q_i)_{i \in V(\Gamma)}$ there is a unique right-angled building $\mathcal{B}$ of type Γ and thickness $\vec{q}$ (up to isomorphism).*

[13]this is the case by [**?**, Proposition 5.34]

For right-angled Artin group we have a nice decomposition or spherical residues. This decomposition will be useful in next section about blow-up buildings. It also gives a sign that the Deligne complex of a right-angled Artin group will be a building (Section 4.5) since there the spherical cosets $gA_\Delta \in A_\Gamma \mathcal{S}^f$ correspond with $\prod_{i\in\Delta}\mathbb{Z}$ structure.

Lemma 4.3.10. *Let $\mathcal{B}$ be a right-angled building of type Γ. Let $\mathcal{R}$ be a spherical residue of a type $J \subseteq V(\Gamma)$ and rank $k := |J|$. Then we can write the residue $\mathcal{R}$ as the product of n many rank 1 residues $\mathcal{R} \cong \prod_{i\in J}\mathcal{R}_i$.*

Proof. We have that $\mathcal{R}$ is a building of type the complete graph of k vertices (since these are the only spherical right-angled Coxeter groups), pick $c_0 \in \mathcal{R}$ a chamber. Consider in this building $\mathcal{R}$ the s_i panels $\mathcal{R}_i$ containing c_0 for each $i \in J$. All these residues $\mathcal{R}_i$ are also buildings (Theorem 4.2.5) consider the following map

$$\begin{aligned}\phi_{c_0} : \mathcal{R} \to \prod_{i\in J}\mathcal{R}_i :\\ c \mapsto \left(proj_{\mathcal{R}_1}(c), proj_{\mathcal{R}_2}(c), \ldots, proj_{\mathcal{R}_k}(c)\right).\end{aligned}$$

We are left to check that this is an isomorphism, i.e. bijective and $c_1 \equiv_{s'} c_2$ if and only if $\phi(c_1)_{s'} \equiv_{s'} \phi(c_2)_{s'}$ and $\phi(c_1)_s = \phi(c_2)_s$ for all $s \in J \setminus \{s_0\}$. For this we refer to [33, Theorem 3.13]. □

Remark 4.3.11. If we have two spherical residues of a right-angled building $\mathcal{R}' \subseteq \mathcal{R}$ (of type J' and J respectively). Then we can always write

$$\mathcal{R} = \mathcal{R}' \times \prod_{i\in J\setminus J'}\mathcal{R}_i,$$

by choosing $c_0 \in \mathcal{R}' \subseteq \mathcal{R}$ in the proof of Lemma 4.3.10. For more information on this see [44, (3.10) theorem]. This way we can also see $\mathcal{R}'$ as a subset of the decomposition $\prod_{i\in J}\mathcal{R}_i$ as follows

$$\mathcal{R}' = \prod_{i\in J'}\mathcal{R}_i \times \prod_{i\in J\setminus J'}\{c_0\}.$$

4.4 Blow-up buildings

Definition 4.4.1 ([33, Definition 5.6])**.** For a right-angled building $\mathcal{B}$, a *blow-up data* is a collection of maps $\mathcal{H} = \{h_\mathcal{R} : \mathcal{R} \to \mathbb{Z} \mid \mathcal{R} \text{ a rank 1 residue of } \mathcal{B}\}$. Such that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel then,

$$h_{\mathcal{R}_2} = h_{\mathcal{R}_1} \circ proj_{\mathcal{R}_1}|_{\mathcal{R}_2} \quad \text{and} \quad h_{\mathcal{R}_1} = h_{\mathcal{R}_2} \circ proj_{\mathcal{R}_2}|_{\mathcal{R}_1}.$$

The *blow-up chamber graph* is the graph with vertex set the set of chambers. Two chambers c and c' are adjacent if they are in the same s-panel $\mathcal{R}$ with $|h_\mathcal{R}(c) - h_\mathcal{R}(c')| \leq 1$.

Lemma 4.4.2 ([33, Lemma 5.7])**.** *Let $\mathcal{R}$ be a spherical residue of type $J \subseteq V(\Gamma)$ such that $\mathcal{R} = \prod_{i\in J} \mathcal{R}_i$ (see Lemma 4.3.10). Suppose there is a blow-up data $\{h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{rank(\mathcal{R})} \mid \mathcal{R}$ a spherical residue$\}$, then we can define map*

$$h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{\text{rank}(\mathcal{R})} \quad \textit{as} \quad h_{\mathcal{R}} := \prod_{i\in J} h_{\mathcal{R}_i} \circ \phi_{c_0}.$$

Moreover, it follows that if $\mathcal{R}$ and $\mathcal{R}'$ are two parallel spherical residues, we have

$$h_{\mathcal{R}} \circ proj_{\mathcal{R}}|_{\mathcal{R}'} = h_{\mathcal{R}'} \quad \text{and} \quad h_{\mathcal{R}'} \circ proj_{\mathcal{R}'}|_{\mathcal{R}'} = h_{\mathcal{R}}.$$

Lemma 4.4.3. *Let $\{h_{\mathcal{R}} \mid \mathcal{R}$ a rank 1 residue$\}$ be a blow-up data of the right-angled building $\mathcal{B}$. Consider two spherical residues $\mathcal{R}' \subseteq \mathcal{R}$. Then there exist a map $h_{\mathcal{R}',\mathcal{R}} : \mathbb{Z}^{rank(\mathcal{R}')} \to \mathbb{Z}^{rank(\mathcal{R})}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{R}' & \xhookrightarrow{\;i\;} & \mathcal{R} \\ \big\downarrow{\scriptstyle h_{\mathcal{R}'}} & & \big\downarrow{\scriptstyle h_{\mathcal{R}}} \\ \mathbb{Z}^{rank(\mathcal{R}')} & \xrightarrow{h_{\mathcal{R}',\mathcal{R}}} & \mathbb{Z}^{rank(\mathcal{R})} \end{array} \tag{4.1}$$

Proof. Choose $c_0 \in \mathcal{R}'$ (as in proof of lemma 4.3.10) then we have

$$\mathcal{R}' = \prod_{i\in J'} \mathcal{R}_i \cong \prod_{i\in J'} \mathcal{R}_i \times \prod_{i\in J\setminus J'} \{c_0\} \subseteq \prod_{i\in J} \mathcal{R}_i = \mathcal{R}.$$

Then define $h_{\mathcal{R}',\mathcal{R}}$ as follows

$$\begin{aligned} h_{\mathcal{R}',\mathcal{R}} : \mathbb{Z}^{\text{rank}(\mathcal{R}')} &\to \mathbb{Z}^{\text{rank}(\mathcal{R})} : \\ &\vec{n} \mapsto \{\vec{n}\} \times \prod_{i\in J\setminus J'} \{h_{\mathcal{R}_i}(c_0)\}. \end{aligned}$$

It is easy to verify that this map satisfies the lemma. □

This notion of a blow-up building will be useful in Section 5.6. However, it has also a very natural meaning, that we will see in next section (See example 4.5.5).

4.5 The Deligne complex of a right-angled Artin group is a building

In this section we answer the question if the Deligne complex (i.e. Geom $\left(\text{flag}\left(A_\Gamma \mathcal{S}^f\right)\right)$ defined in Definition 2.10.2) of an Artin group is a building $\mathcal{B}$, in the sense that the chambers of $\mathcal{B}$ correspond to the elements of A_Γ and two chambers $v, w \in A_\Gamma$ are s-adjacent (for $s \in V(\Gamma)$) in $\mathcal{B}$ if and only if $vA_{\{s\}} = wA_{\{s\}}$. The reason we want this to be to meaning of "the Deligne complex is a building" is, since then we have precisely that two chambers are in the same $\mathcal{R}$ spherical residue of type

Δ (a subgraph of Γ) if and only if they are contained in the same spherical coset in $A_\Gamma \mathcal{S}^f$ of the form gA_Δ. Just as it was the case for $\mathcal{B}_{W_\Gamma}$ and the Davis complex $W_\Gamma \mathcal{S}^f$. Also, it will make sure the Moussong metric on the Deligne complex (Definition 2.11.7) coincides with the CAT(0) metric on the building.

One would think this is the case because the Davis complex is a building and the Deligne complex is build out of the Artin group the same way the Davis complex is build out if the Coxeter group. Even more every vertex in the Deligne complex is contained in a subcomplex isomorphic with the Davis complex, and the Davis complex is exactly one apartment. Unfortunately in general it will not be a building.

Example 4.5.1. (i) Consider the Artin group of type $\Gamma := \overset{a}{\bullet}\overset{3}{\text{---}}\overset{b}{\bullet}$. Since the maximal length of elements in W_Γ is 3, the distance between two chambers cannot be bigger than 3. However, suppose a building $\mathcal{B}$ exists satisfying the above, then one can check that ab^{-1} and ba^{-1} have distance at least 4. So $A_\Gamma \mathcal{S}^f$ will not be a building.

(ii) Now let A_Γ be the right-angled Artin group of type $\Gamma := \overset{a}{\bullet}\overset{2}{\text{---}}\overset{b}{\bullet}$. With some abuse of notation the complex in Figure 2.10.1 of Construction 2.10.3 is the Deligne complex of A_Γ. One can check this forms a building of type Γ for the following map

$$\delta : A_\Gamma \times A_\Gamma \to W_\Gamma : (v,w) \mapsto \begin{cases} \mathbb{1} & \text{if } v =_{A_\Gamma} w; \\ a & \text{if } v \neq_{A_\Gamma} w \text{ and } vA_{\{a\}} = wA_{\{a\}}; \\ b & \text{if } v \neq_{A_\Gamma} w \text{ and } vA_{\{b\}} = wA_{\{b\}}; \\ ab(=ba) & \text{otherwise.} \end{cases}$$

This map coincides with $(v,w) \mapsto (v^{-1}w)_{W_\Gamma}$, here we mean with $(\cdot)_{w_\Gamma}$ the value of this element if W_Γ (i.e. the image under te map $A_\Gamma \to W_\Gamma$).

For right-angled buildings this will be the case in general. We first notice one important remark.

Remark 4.5.2. The set of spherical cosets $A_\Gamma \mathcal{S}^f$ (i.e. cosets of spherical subgroups see Definition 1.1.6) of a right-angled Artin group are the cosets of the abelian subgroups $A_\Lambda (=\mathbb{Z}^{|V(\Lambda)|})$ where Λ spans a clique in Γ.

Theorem 4.5.3. *Let A_Γ an arbitrary right-angled Artin group and let $V(\Gamma) = \{s_1, s_2, \ldots, s_n\}$, then there is a building $\mathcal{B}$ of type Γ satisfying the following:*

(i) *The Chambers of $\mathcal{B}$ correspond with the elements of A_Γ.*

(ii) *Two chambers $c_1, c_2 \in \mathcal{B}$ are s_i-adjacent if and only if $c_1 A_{\{s_i\}} = c_2 A_{\{s_i\}}$.*

(iii) *We have that* $\mathrm{Geom}(\mathcal{B}) \cong \mathrm{Geom}\big(\mathrm{Flag}(A_\Gamma \mathcal{S}^f)\big)$.

(iv) *A rank k spherical residue in $\mathcal{B}$, correspond with a coset gA_Λ in $A_\Gamma\mathcal{S}^f$ such that $|V(\Lambda)| = k$. Moreover, the spherical residues that contain a given element $g \in A_\Gamma$ are precisely the cosets gA_Λ where $\Lambda \underset{clique}{\subseteq} \Gamma$.*

Proof. Let $X_s := A_{\{s\}} \cong \mathbb{Z}$ in Construction 4.3.7, then $G_\Gamma\left((A_{\{s\}})_{s\in S}\right) = A_\Gamma$. From Theorem 4.3.8 we have that $\Phi\left(A_\Gamma, \{\mathbb{1}\}, (A_{\{s\}})_{s\in S}\right)$ is a building. One can easily verify that it satisfies the statement. See also [23, Theorem 5.1]. □

Definition 4.5.4. Let A_Γ be a right-angled Artin group then the *associated building* denoted by $\mathcal{B}_{A_\Gamma}$ is the building form Theorem 4.5.3.

Example 4.5.5. Consider the following building $\mathcal{B}$ of type $\Gamma := \overset{a}{\bullet}\overset{2}{\longrightarrow}\overset{b}{\bullet}$. The chamber are the elements of $\mathbb{Z} \times \mathbb{Z} = A_\Gamma$. The set $\mathbb{Z} \times \mathbb{Z}$ can be visualized as a grid (by looking at its Cayleygraph, also see Figure 2.10.1). We define two equivalence relations; two vertices are a-equivalent if they lay on the same vertical line and b-equivalent if they are on the same horizontal line. Every vertical line is an a-residue (or a-panel) and every horizontal line is a b-residue (all in Figure 2.10.1). There is only one rank 2 residue and that is the whole plane containing every point on the grid. In Figure 2.10.1 we see that the lines (rank 1 residues) correspond with the cosets of A_Γ. In this way we visualize the building $\mathcal{B}$ as a grid. This is actually the way a blow-up building works. Let gA_a and hA_b be arbitrary cosets of rank 1, the maps of the blow-up building are

$$h_{b^iA_a} : b^iA_a \to \mathbb{Z} : b^ia^n \mapsto n;$$
$$h_{a^iA_b} : a^iA_b \to \mathbb{Z} : a^ib^n \mapsto n.$$

In this case al rank 1 residues of the same type are parallel (this is not to case in general not for arbitrary right-angled building nor for arbitrary buildings constructed in Theorem 4.5.3), with the following projection maps

$$proj_{b^jA_a}(b^iA_a) : b^iA_a \to b^jA_a : b^ia^n \mapsto b^ja^n;$$
$$proj_{a^jA_b}(a^iA_b) : a^iA_b \to a^jA_b : a^ib^n \mapsto a^jb^n.$$

4.6 Non-right-angled Artin groups are never Buildings

Unfortunately, for non-right-angled Artin groups the Deligne complex will never be a building. This is partially the case since that for right-angled Artin groups we have that $ab = ba \Rightarrow a^{-1}b = ba^{-1}$ while in the non-right-angled case $abab\cdots = baba\cdots \not\Rightarrow a^{-1}ba^{-1}b\cdots = ba^{-1}ba^{-1}\cdots$. We first prove the following.

Lemma 4.6.1. *Let A_Γ an Artin group of type $\Gamma := \overset{a}{\bullet}\overset{n}{\longrightarrow}\overset{b}{\bullet}$ with $n > 2$. Then A_Γ cannot be written as a finite product of the subgroup A_a and A_b, i.e.*

$$\forall m \in \mathbb{N},\ A_\Gamma \neq \underbrace{A_aA_bA_aA_a\ldots}_{m\ sets}. \tag{4.2}$$

Before proving this we discuss this in general, a group G is *boundedly generated* if there is a finite set of elements $S := \{s_1, s_2, \dots, s_m\} \subseteq G$ such that

$$G = \{s_1^{k_1} s_2^{k_2} \cdots s_m^{k_m} \mid s_i \in S, k_i \in \mathbb{N}\}.$$

It is known that free groups (of rank ≥ 2) are not boundedly generated. Moreover, a finite index subgroup $H \underset{\text{finite index}}{\leq} G$ is boundedly generated if and only if G is. We also need the following lemma.

Lemma 4.6.2 ([4, Lemma 2.5])**.** *Let A_Γ be an Artin group of type $\Gamma := \overset{a}{\bullet}\overset{n}{\text{———}}\overset{b}{\bullet}$ with $n > 2$. Then A_Γ contains a finite index normal subgroup of the form $\mathbb{Z} \times F_w$ with F_w a free group of rank w. Moreover, $A_\Gamma / (\mathbb{Z} \times F_w) \cong \mathbb{Z}/w\mathbb{Z}$.*

Proof of Lemma 4.6.1. It is sufficient to prove that A_Γ is not boundedly generated. By lemma 4.6.2, we know that A_Γ contains a finite index subgroup $\mathbb{Z} \times F_w$, if $w = 1$ then we would have $A_\Gamma \cong \mathbb{Z} \times \mathbb{Z}$, which is a contradiction (since, $n > 2$). Hence, $w \geq 2$. We prove that if $\mathbb{Z} \times F_w$ is boundedly generated, then so is F_w. Suppose $\forall (l, r) \in \mathbb{Z} \times F_w$ we have

$$(l, r) = (l_1, r_1)^{k_1} (l_2, r_2)^{k_2} \cdots (l_m, r_m)^{k_m},$$

for a fixed $m \in \mathbb{N}$. But then clearly every $l \in F_w$ can be written as

$$l = l_1^{k_1} l_2^{k_2} \cdots l_m^{k_m},$$

hence F_w is boundedly generated. Since this is a contradiction we know that $\mathbb{Z} \times F_m$ could not have been boundedly generated. Since $\mathbb{Z} \times F_m \underset{\text{finite index}}{\leq} A_\Gamma$, A_Γ is also not boundedly generated. $\square$

Clearly the previous proof does not work if $n = 2$, since then $A_\Gamma = A_a A_b$. We have now enough information to prove that non-right-angled Artin groups are not buildings.

Remark 4.6.3. One important remark is that the "distance" in the group A_Γ with $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$ between elements is the amount of generators we need, for example $a^2 b^{-5}$ and $\mathbb{1}$ have distance 7 between them. However, in a building $\mathcal{B}_{A_\Gamma}$ the distance is only two, since $\mathbb{1} \equiv_a a^2 \equiv_b a^2 b^{-5}$.

Theorem 4.6.4. *Let A_Γ an Artin group that is not right-angled, then there does not exist a building $\mathcal{B}$ such the chambers are the elements of A_Γ and two chambers $g, h \in A_\Gamma$ are s-equivalent (with $s \in V(\Gamma)$) if and only if $g \in hA_{\{s\}}$.*

Proof. Suppose there is a building $\mathcal{B}$ satisfying the asked. Since A_Γ is not right-angled, there exist a subgraph of the form $\overset{a}{\bullet}\overset{n}{\text{———}}\overset{b}{\bullet} =: \Lambda \subseteq \Gamma$, where $n > 2$. Consider x an element in A_Γ that is not contained in $\underbrace{A_a A_b \cdots}_{2n}$ (this exists by

Lemma 4.6.1). We will prove that the distance in the alleged building $\mathcal{B}$ is at least $2n+1$, in our proof it will follow from Theorem 4.2.5. Clearly both x and $\mathbb{1}$ are contained in the same residue $\mathcal{R}_\Lambda$ of type Λ. Hence, by Theorem 4.2.5 (i) we have $\delta_{\mathcal{B}}(x, \mathbb{1}) \in W_\Lambda$, however, the maximal length between them can then only be n, since the elements of W_Λ have at most length n. However, the size of the minimal gallery between them in the building $\mathcal{R}_\Lambda (= A_\Lambda)$ is at least $2n+1$ (by choice of x). This is a contradiction. $\square$

In Lemma 4.6.1 we maybe did some "overkill" to prove that there are elements in A_Γ (for $\Gamma := \overset{a}{\bullet}\overset{n}{\text{———}}\overset{b}{\bullet}$ with $n > 2$) that are at least $n+1$ distance (as in equivalences in the chamber system) of $\mathbb{1}$. One can probably do this easier by picking $x := \underbrace{ab^{-1}ab^{-1}a\cdots}_{n+1 \text{ terms}}$ in the proof of Theorem 4.6.4. However, then you first need to prove that x cannot be written as a product with fewer alternating generators, which is maybe more difficult than it looks. Other words like $abababa\cdots$ cannot be uses, for example $ababa = a^2ba^2$ if $\Gamma := \overset{a}{\bullet}\overset{3}{\text{———}}\overset{b}{\bullet}$. What we actually proved in Theorem 4.6.4 is that the chamber system $\Phi\left(A_\Gamma, \{\mathbb{1}\}, (A_s)_{s\in V(\Gamma)}\right)$ is not a building.

5 Constructing building from exploded Salvetti complex

In the upcoming Chapter we will only work with right-angled Artin groups. We will define the exploded Salvetti complex. This complex will give us an easy way to go to the right-angled building associated to the Artin group. It will also be useful in the quasi-isometric classification of right-angled Artin groups. This complex was first introduced in [9] for A_Γ two-dimensional (i.e. Γ does not contain cliques of more than two vertices) and in [33] by Huang and Kleiner for all right-angled Artin groups. There will be a map from the exploded Salvetti complex to the right-angled building, this map will turn out to be a restriction quotient map, which we will define now.

5.1 Restriction quotient maps

For a CAT(0) cube complex there is a natural notion of a hyperplane, this is a subspace that "cuts" the cube complex in two parts. For this we first define it for regular cubes.

Definition 5.1.1. A *midcube* of on n-dimensional cube $[0,1]^n$ is s subset of the form

$$[0,1]^{i-1} \times \left\{\frac{1}{2}\right\} \times [0,1]^{n-i} \subseteq [0,1]^n.$$

Hence, an n-dimensional cube has n many midcubes.

Definition 5.1.2. Let X be a CAT(0) cube complex. Consider the following relation between edges of X. Let $e \sim e'$ if they are opposite a 1 cube. Then consider the equivalence relation on the edges of X generated by the relation $\sim$. Denote the equivalence class of an edge by $[e]$ (i.e. $[e]$ is the set of edges such that for every edge $e' \in [e]$ there is a sequence of edges $e = e_1, e_2, \dots, e_k = e'$ such that e_i and e_{i+1} are opposite one another in a 1 cube). A *hyperplane H dual to an edge* $e_0 = [0,1] \subseteq X$ is the union of midcubes $H := \cup_i M_i$ such that every of these midcudes intersect at least one edge $e \in [e_0]$. We denote the set of hyperplanes of X by $\hat{\mathcal{H}}$. If a hyperplane H intersect an edge e we call H *dual* to e.

For CAT(0) cube complexes this notion of hyperplanes behaves nicely.

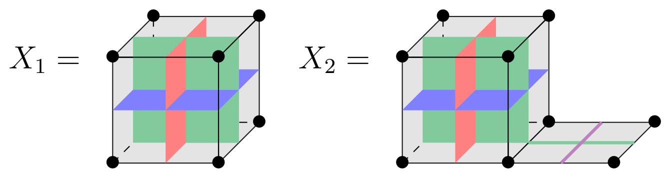
Theorem 5.1.3 ([45, Theorem 1.1]). *Let X be a* CAT(0) *cube complex. Then the following hold.*

(i) *For every hyperplane H the inclusion $H \to X$ is injective.*

(ii) *Every hyperplane separates X in exactly two components.*

(iii) *Every hyperplane is a* CAT(0) *cube complex.*

Example 5.1.4. Consider the following cube complexes.



The cube complex X_1 has 3 hyperplanes, the cube complex X_2 has one more.

From Theorem 5.1.3 (ii) we can define halfspaces.

Definition 5.1.5. Let X be a CAT(0) cube complex with the set of hyperplanes being $\hat{\mathcal{H}}$. A *halfspace* is the closure of one of the two components of X that is separated by a hyperplane. We denote the set of halfspaces by $\mathcal{H}$. For a subset of hyperplanes $\hat{\mathcal{K}} \subseteq \hat{\mathcal{H}}$ we also denote $\mathcal{K}$ to be the set of halfspaces that are separated by hyperplanes of $\hat{\mathcal{K}}$.

We will see that a CAT(0) cube complex is uniquely determined by its set of hyperplanes. This will the be useful to define special maps between CAT(0) cube complex (which will be called restriction quotient maps), they will be determined from the moment we establish which hyperplanes we keep and which we forget. It will turn out that the set of halfspaces will form a "pocset".

Definition 5.1.6 ([43, page 4]). A *pocset* $(P, \leq, *)$ is a poset with an involution map $* : P \to P$, that satisfy; for every $A \in P$ we have that A^* is not comparable to A (i.e. $A \nleq A^*$ and $A \ngeq A^*$). A pocset has *finite interval condition* if for every pair $A, B \in P$ such that $A \subseteq B$ we have $\#\{C \in P \mid A \subseteq C \subseteq B\}$ is finite.

Lemma 5.1.7. *For a* CAT(0) *cube complex X with the set of hyperplanes $\hat{\mathcal{H}}$, the set of halfspaces forms a pocset for the inclusion and for the following involution map:*

$$\overline{(\cdot)^c} : \mathcal{H} \to \mathcal{H} :$$
$$A \mapsto \overline{A^c} \text{ i.e. the closure of the complement in } X.$$

Proof. Exercise. □

Theorem 5.1.8 (Sageev & Roller [16, Section 2.3])**.** *For every pocset* $(P, \leq, *)$ *(with finite interval condition) there is a unique* CAT(0) *cube complex such that the pocset* $(\mathcal{H}, \subseteq, \overline{(\cdot)^c})$ *isomorphic is with* $(P, \leq, *)$*.*

Sketch of proof. We give the idea of how one would construct a cube complex from a pocset. For every ultrafilter $\mathcal{F} \subseteq P$ we have one point. We draw an 1-cube between two points if they differ by only one element. One then attaches an n-cube for every time we see an n-cube in the 1-skeleton. For a detailed description see [16, Section 2.3] and [45, Section 2]. □

By Theorem 5.1.8 we can reconstruct our CAT(0) cube complex by just knowing the set of Hyperplanes. Having this we can define the following.

Definition 5.1.9 (Restriction quotient map)**.** A map $q : X \to Y$ between two CAT(0) cube complexes is a *restriction quotient map* if it can be constructed as follows: there exist a subset $\mathcal{K} \subseteq \mathcal{H}_X$ of halfspaces that is closed under taking $\overline{(\cdot)^c}$ such that $Y \cong X(\mathcal{K})$ and

$$q : X \to X(\hat{\mathcal{K}}) : \mathcal{F} \mapsto \mathcal{F} \cap \mathcal{K},$$

here we identified a vertex of a cube complex with its ultrafilter (see proof of Theorem 5.1.8). This can then be extended to a surjective cubical map (for more information [33, Definition 4.1]).

Example 5.1.10. Consider the following cube complex.

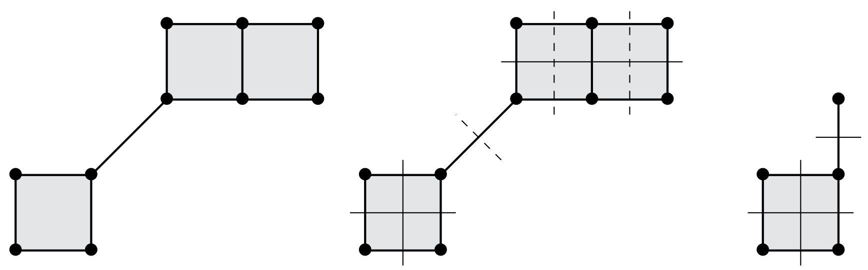


Figure 5.1.1: Cube complex and its hyperplanes

This cube complex has 6 hyperplanes, and we denote $\hat{\mathcal{K}}$ to be the set of hyperplanes in Figure 5.1.1 that are not dashed (here $|\mathcal{K}| = 3$). Then the restriction quotient map $q : X(\mathcal{H}) \to X(\mathcal{K})$, send X to the cube complex on the right side in Figure 5.1.1.

We will need the following lemma in Section 5.7.

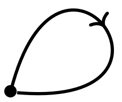
Lemma 5.1.11 ([33, Lemma 4.3])**.** *Let* X *be a* CAT(0) *cube complex. Let* $q : X\left(\hat{\mathcal{H}}_X\right) \to X\left(\hat{\mathcal{K}}\right)$ *a restriction quotient map, with* $\hat{\mathcal{K}}$ *be a subset of the hyperplanes of* X*. Let* $\alpha : X \to X$ *be a cubical isomorphism such that for every* *edge $e \subseteq X$; e is dual to a hyperplane of $\hat{K}$ if and only if $\alpha(e)$ is dual to a hyperplane of $\hat{K}$. Then α induces an isomorphism $X\left(\hat{K}\right) \to X\left(\hat{K}\right)$.*

5.2 Exploded Salvetti complex

Like the Salvetti complex which was a graph product of circles, the exploded Salvetti complex will be a graph product of a space that is homotopy equivalent to a circle (it will be a lollipop ☺ (see [33, page 544])). We first give the definition of the Salvetti complex for right-angled Artin groups again as a reminder.

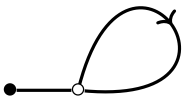
Definition 5.2.1 (Salvetti complex)**.** The *Salvetti complex* $\mathcal{S}_\Gamma$ of a right-angled Coxeter group W_Γ is the Γ-graph product constructed if for every $v \in V(\Gamma)$ we use $(X_v, p_v) := (S^1, (0,1))$ in Definition 2.9.1. Hence,

$$\mathcal{S}_\Gamma := \widetilde{\prod}_{v \in V(\Gamma)} (S^1_v, \bullet_v) = \bigcup_{\substack{\Delta \subseteq V(\Gamma) \\ \text{clique}}} \left(\prod_{v \notin \Delta} \{\bullet_v\} \times \prod_{v \in \Delta} (S^1_v, \bullet_v) \right),$$

where $S^1 =$  .

Definition 5.2.2 (Exploded Salvetti complex)**.** Let L be the topological space consisting of one circle S^1 and one line of length one (isomorphic to $[0,1] \subseteq \mathbb{R}$) attached to one point of the circle. We denote $\bullet$ to be the endpoint of $[0,1]$ that has degree 1 in L. Let A_Γ be an Artin group of type Γ, then the *exploded Salvetti complex* denoted by $\mathcal{S}^e_\Gamma$ is the Γ-graph product

$$\mathcal{S}^e_\Gamma := \widetilde{\prod}_{v \in V(\Gamma)} (L_v, \bullet_v) = \bigcup_{\substack{\Delta \subseteq V(\Gamma) \\ \text{clique}}} \left(\prod_{v \notin \Delta} \{\bullet_v\} \times \prod_{v \in \Delta} (L^1_v, \bullet_v) \right),$$

where $L_v :=$  . We will also denote this free endpoint by $\bullet_v$, and the point connected with the circle by $\circ_v$.

Example 5.2.3. (i) Let $\Gamma := \overset{a}{\bullet} \quad \overset{b}{\bullet}$, then Figure 5.2.1 shows the Salvetti complex and the exploded Salvetti complex.

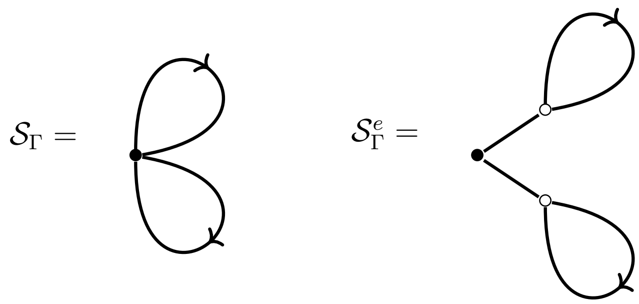


Figure 5.2.1: The Salvetti and exploded Salvetti complex of Γ.

(ii) Now consider $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$. Then the Salvetti complex $(S^1_a \times S^1_b)$ and the Exploded Salvetti complex $(L_a \times L_b)$ are drawn in Figure 5.2.2.

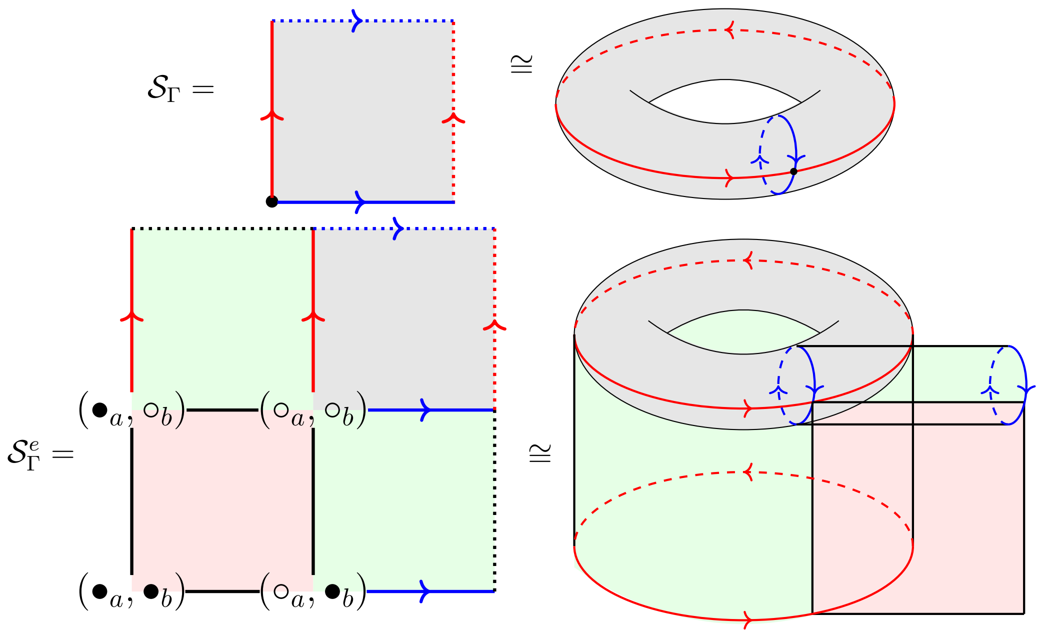


Figure 5.2.2: The Salvetti complex and the exploded Salvetti complex of type Γ.

Here the Salvetti complex is just a torus, the exploded Salvetti complex is a torus $(S^1 \times S^1)$ such that the two spanning loops extend to two cylinders $(S^1 \times [0,1])$ that are themselves attached via a $[0,1] \times [0,1]$ plane.

(iii) The exploded Salvetti complex of type $\Gamma :=$ $\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$ is drawn on the front page of this thesis.

Definition 5.2.4. Let A_Γ be a right-angled Artin group of type Γ,

(i) We denote the universal cover of the Salvetti complex by $\overline{\mathcal{S}_\Gamma}$ and the universal cover of the exploded Salvetti complex by $\overline{\mathcal{S}^e_\Gamma}$.

(ii) Let $\phi : \overline{\mathcal{S}_\Gamma} \to \mathcal{S}_\Gamma$ and $\phi_e : \overline{\mathcal{S}^e_\Gamma} \to \mathcal{S}^e_\Gamma$ be the respective covering maps. The natural continuous map $\tau : \mathcal{S}^e_\Gamma \to \mathcal{S}_\Gamma$ that retracts the $[0,1]$ parts to a point is a homotopy equivalence (i.e. it extends to an isomorphism τ^* between $\pi_1(\mathcal{S}^e_\Gamma)$ and $\pi_1(\mathcal{S}_\Gamma)$). By definition of covering map (see also Lemma 6.9.3) we can lift this map to a map $\overline{\tau} : \overline{\mathcal{S}^e_\Gamma} \to \overline{\mathcal{S}_\Gamma}$ such that the following diagram commutes.

$$\begin{array}{ccc} \overline{\mathcal{S}^e_\Gamma} & \xrightarrow{\overline{\tau}} & \overline{\mathcal{S}_\Gamma} \\ \big\downarrow{\scriptstyle \phi_e} & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{S}^e_\Gamma & \xrightarrow{\tau} & \mathcal{S}_\Gamma \end{array} \tag{5.1}$$

Remark 5.2.5. There is only one "vertex" in $\mathcal{S}_\Gamma$ that being $(\bullet_v)_{v\in V(\Gamma)}$ (See also Definition 2.9.15), that we also denote as just $\bullet$. Consider now the following points in $\mathcal{S}^e_g$:

$$(*_v)_{v\in V(\Gamma)}, \text{ where } *_v = \begin{cases} \bullet & \text{if } v \in V(\Gamma)\setminus\Delta, \\ \circ & \text{if } v \in \Delta, \end{cases}$$

where Δ spans clique in Γ. There are in total $|\mathcal{S}^f|$ of these points in $\mathcal{S}^e_\Gamma$. Indeed, every spherical subgroup of $A_\Delta \le A_\Gamma$ correspond with one unique clique Δ in Γ, now by Definition 2.9.1 we see that every clique correspond with a unique product and each product correspond with a unique choice of $\circ_v$'s. A point of $\mathcal{S}^e_\Gamma$ thus correspond to a set $\Delta \in \mathcal{S}^f$. Hence, we can associate one unique k-torus for every point $(*_v)_{v\in V(\Gamma)}$ in $\mathcal{S}^e_\Gamma$, where k is the number of indices $v \in V(\Gamma)$ such that $*_v = \circ$. This k-torus is the following

$$\prod_{*_v=\circ}(S^1_v, \circ_v) \times \prod_{*_v=\bullet_v}\{\bullet_v\} = \prod_{v\in\Delta}(S^1_v, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}.$$

Definition 5.2.6. (i) Both the Salvetti complex and the exploded Salvetti complex are cube complexes. Since they are a product of intervals $[0,1]$ and circles S^1 which are both 1-cubes, where in the first case there is a 1 cube glued on two different 0-cubes (or vertices) and in the second case to the same vertex. The set of vertices of $\mathcal{S}_\Gamma$ correspond to the singleton $\{(\bullet_v)_{v\in V(\Gamma)}\}$, the vertices of $\mathcal{S}^e_\Gamma$ are these constructed in Remark 5.2.5.

(ii) We will call the elements in $\phi^{-1}(\bullet)$ vertices in $\overline{\mathcal{S}_\Gamma}$. The elements in $\phi_e^{-1}\big((*_v)_{v\in V(\Gamma)}\big)$ vertices of $\overline{\mathcal{S}^e_\Gamma}$, where these vertices $(*_v)_{v\in V(\Gamma)}$ are constructed in Remark 5.2.5.

(iii) If we talk about a *point* in the cube complexes $\mathcal{S}_\Gamma, \mathcal{S}^e_\Gamma, \overline{\mathcal{S}_\Gamma}, \overline{\mathcal{S}^e_\Gamma}$, instead of a vertex we mean an arbitrary point in these metric spaces (possibly in the interior of a cube) not necessarily a 0-cube.

Example 5.2.7. We work further on Example 5.2.3.

(i) We had that $\Gamma := \overset{a}{\bullet} \quad \overset{b}{\bullet}$. The exploded Salvetti complex contains 3 vertices being $(\bullet_a, \bullet_b), (\circ_a, \bullet_b)$ and $(\bullet_a, \circ_b)$. They are contained in the tori $(\bullet_a, \bullet_b), (S^1_a, \bullet_b)$ and $(\bullet_a, S^1_b)$ respectively.

(ii) We had that $\Gamma := \overset{a}{\bullet}\!\!-\!\!-\!\!\overset{b}{\bullet}$. The exploded Salvetti complex contains 4 vertices being $(\bullet_a, \bullet_b), (\circ_a, \bullet_b), (\bullet_a, \circ_b)$ and $(\circ_a, \circ_b)$. They are contained in the tori $(\bullet_a, \bullet_b), (S^1_a, \bullet_b), (\bullet_a, S^1_b)$ and (S^1_a, S^1_b) respectively.

Since we will be working with liftings of path in the fundamental groups $\pi_1(\mathcal{S}_\Gamma)$ and $\pi_1(\mathcal{S}^e_\Gamma)$, it is useful to chose fixed basepoints.

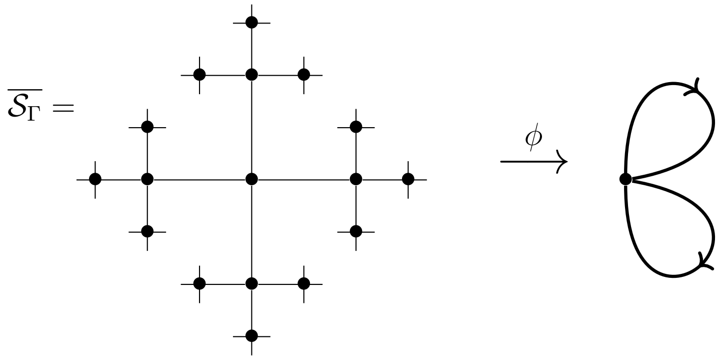
Definition 5.2.8. By Theorem 2.9.17 we have that $\pi_1(\mathcal{S}_\Gamma) = A_\Gamma$. One can easily see that also $\pi_1(\mathcal{S}^e_\Gamma) = A_\Gamma$. From now on we will always choose that same basepoint in these fundamental groups, that being $(\bullet_v)_{v \in V(\Gamma)} =: \bullet$ for the Salvetti complex and exploded Salvetti complex, and thus write $\pi_1(\mathcal{S}_\Gamma, \bullet)$ and $\pi_1(\mathcal{S}^e_\Gamma, \bullet)$. We will also choose fixed basepoints in the universal covers. Pick a vertex $\bullet^e_0 \in \phi_e^{-1}\left((\bullet_v)_{v \in V(\Gamma)}\right)$, and $\bullet_0 \in \overline{\tau}(\bullet^e_0)$.

Remark 5.2.9. (i) Since Diagram (5.1) commutes it is easy to see that the chosen basepoint $\bullet_0 \in \overline{\tau}(\bullet^e_0)$ is contained in $\phi^{-1}(\bullet)$.

(ii) Since $\overline{\mathcal{S}_\Gamma}$ is the universal cover of $\mathcal{S}_\Gamma$, take basepoint $\bullet_0 \in \overline{\mathcal{S}_\Gamma}^{(0)} = \phi^{-1}(\bullet)$, then each path $p \in \pi_1(\mathcal{S}_\Gamma, \bullet)$ lifts to a path $\tilde{p} \subseteq \overline{\mathcal{S}_\Gamma}$ starting at $\bullet_0$ with a unique endpoint $\bullet_1$. We also have for every point $\bullet_1 \in \overline{\mathcal{S}_\Gamma}^{(0)}$, there is a unique element $a \in A_\Gamma = \pi_1(\mathcal{S}_\Gamma, \bullet)$ that lifts to a path in $\overline{\mathcal{S}_\Gamma}$ starting at $\bullet_0$ and ending at $\bullet_1$ (for a proof, see [29, Theorem 7.4]). Similarly, for $\overline{\mathcal{S}^e_\Gamma}$.

(iii) Every hyperplane of the Salvetti complex is isomorphic to the Salvetti complex of a subgroup (also see [45, Exercise 1.4]). The same holds for the exploded Salvetti complex.

Example 5.2.10. Let $\Gamma := \overset{a}{\bullet} \quad \overset{b}{\bullet}$, then the universal covers $\overline{\mathcal{S}_\Gamma}$, $\overline{\mathcal{S}^e_\Gamma}$ and covering maps are drawn in Figure 5.2.3.



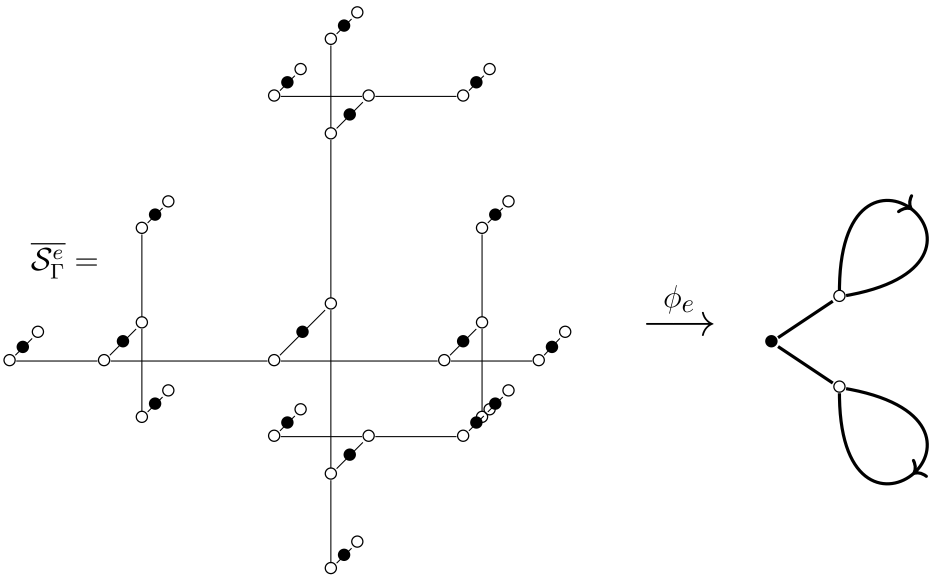


Figure 5.2.3: Universal covers of $\mathcal{S}_\Gamma$ and $\mathcal{S}^e_\Gamma$.

We already saw in Theorem 3.3.10 **(2)** that $\overline{\mathcal{S}_\Gamma}$ is a CAT(0) space. This is also true for the universal cover of the exploded Salvetti complex (see also [33, Page 567]). First we have the following.

Lemma 5.2.11 ([33, page 3])**.** *The Salvetti complex and the exploded Salvetti complex of right-angled Artin groups are non-positively curved cube complexes*

Proof. By property 2.3.9. □

Theorem 5.2.12. *The universal cover $\overline{\mathcal{S}^e_\Gamma}$ of the exploded Salvetti complex is a* CAT(0) *space.*

Proof. By Property 2.3.9, is $\mathcal{S}^e_\Gamma$ a non-positively curved cube complex space. By Theorem 3.1.4 the universal cover $\bar{\mathcal{S}_\Gamma}$ is CAT(0). □

Both the Salvetti complex and the exploded Salvetti complex are non-positively curved. However, they are both not CAT(0) since this would be in contradiction with Theorem 2.3.7, that states that they are contractible while they are both not even simply connected.

5.3 Flats in the universal cover of the exploded Salvetti complex

In this section we will define standard flats in the universal cover of the Salvetti complex, these subspaces will be in one to one correspondence with the set of spherical cosets $A_\Gamma \mathcal{S}^f$ of our Artin group, and will be important to the link with the associated right-angled building, which Theorem 4.5.3 (iii) already suggests.

Definition 5.3.1. For A_Γ an Artin group, let $\Delta \subseteq V(\Gamma)$ with $|\Delta| = k$ such that Δ spans a clique in Γ.

(i) A subcomplex of $\mathcal{S}_\Gamma$ is a *standard k-torus* if it is of the following form

$$\prod_{v \in \Delta} (S^1_v, \bullet_v) \times \prod_{v \in V(\Gamma) \setminus \Delta} \{\bullet_v\},$$

which is a subcomplex consisting of a k-torus.

(ii) A subcomplex of $\mathcal{S}^e_\Gamma$ is a *standard k-torus* if it is of the following form

$$\prod_{v \in \Delta} (S^1_v, \circ_v) \times \prod_{v \in V(\Gamma) \setminus \Delta} \{\bullet_v\},$$

which is a subcomplex consisting of a k-torus, where $S^1_v \subseteq L_v$.

For every clique $\Delta \subseteq V(\Gamma)$ we will denote the T_Δ for the standard $|\Delta|$-torus (from the context it then needs to be clear if we mean a subcomplex of $\mathcal{S}_\Gamma$ or $\mathcal{S}^e_\Gamma$).

Definition 5.3.2. A *standard k-flat* is a connected component of the inverse image by ϕ of a standard k-torus T_Δ for e certain clique Δ. In the context of the exploded Salvetti complex a *standard k-flat* is a connected component of the inverse image under ϕ_e of a standard k-torus T_Δ. We also call a 1-flat a *standard geodesic*[14].

Definition 5.3.3. A standard flat F in $\overline{\mathcal{S}_\Gamma}$ (or in $\overline{\mathcal{S}^e_\Gamma}$) has *type* Δ and has *rank* $|\Delta|$ if $\phi(F) = T_\Delta$ (or $\phi_e(F) = T_\Delta$).

We will later prove that there is a bijection (Theorem 5.3.16) between these standard flats and the spherical cosets of A_Γ / spherical residues of the building $\mathcal{B}_{A_\Gamma}$.

Definition 5.3.4. Let X be a CAT(0) cube complex with metric $d(\cdot,\cdot)$. Consider two convex subsets $C_1, C_2 \subseteq X$. They are parallel if the maps $d(\cdot, C_1)|_{C_2}$ and $d(\cdot, C_2)|_{C_1}$ are constant.

Example 5.3.5. We continue where we left off in Example 5.2.3 (ii) where we had $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$. Both for the Salvetti complex as for the Exploded Salvetti complex, there are 4 standard tori. In the Exploded Salvetti complex they are all disjoint (with distance 1 between them), in the Salvetti complex they are never disjoint.

- The standard tori of $\mathcal{S}_\Gamma$ in Figure 5.2.2 is the vertex $\bullet$, the red and blue loops and the gray torus.

[14]One should be careful with this definition since here a standard geodesic will always be isomorphic to the line $\mathbb{R}$, while $\mathbb{R}$ can never be the image of geodesic as in Definition 2.3.1.

- The standard tori of $\mathcal{S}^e_\Gamma$ in Figure 5.2.2 is the vertex $(\bullet, \bullet)$, the red loop starting at $(\bullet, \circ)$, the blue loop starting at $(\circ, \bullet)$ and the torus with base point $(\circ, \circ)$.

The standard flats for this example are explained later in Example 5.3.17.

Remark 5.3.6. (i) Let $F \subset \overline{\mathcal{S}_\Gamma}$ be a flat and $\phi(F) = T_\Delta$. Since F is by definition a connected component of $\phi^{-1}(T_\Delta)$, it universally covers T_Δ. Hence, since the universal cover of a $(|\Delta| =: k)$-torus is isomorphic to $\mathbb{R}^k$, we have that $F \cong \mathbb{R}^k$. Similarly, if $F \subseteq \overline{\mathcal{S}^e_\Gamma}$ is a flat with $\phi_e(F) = T_\Delta$, then $F \cong \mathbb{R}^k$. However, we will often look at a flat in the sense of a cube complex with 0-skeleton $\mathbb{Z}^k$.

(ii) Let F be a standard k-flat of $\overline{\mathcal{S}^e_\Gamma}$, then $\overline{\tau}(F) =: F'$ is a k-flat in $\overline{\mathcal{S}_\Gamma}$. This is the case since Diagram (5.1) commutes and because $\overline{\tau}$ is continuous it maps connected spaces to connected spaces. However, F is **not** the inverse image of F' under $\overline{\tau}$ (i.e. $F \subsetneq \overline{\tau}^{-1}(F')$). More precisely, $\tau^{-1}(F')$ will be isomorphic to $F' \times [0,1]^{m-k} \cong F \times [0,1]^{m-k}$, where m is the dimension of the maximal dimensional torus containing the torus $\phi(F')$ and k the dimension the torus $\phi(F')$.

Definition 5.3.7. Suppose a group G has an action on two spaces X and Y. Then a map $\tau : X \to Y$ is *equivariant* if for all points $x \in X$ and elements $g \in G$ we have $\tau(v^g) = \tau(v)^g$.

Lemma 5.3.8. *There is an action of A_Γ on both $\overline{\mathcal{S}^e_\Gamma}$ and $\overline{\mathcal{S}_\Gamma}$, such that $\overline{\tau} : \overline{\mathcal{S}^e_\Gamma} \to \overline{\mathcal{S}_\Gamma}$ is an equivariant map.*

Proof. **Step 1: Define action**: By Definition 5.2.4 (ii) we have an isomorphism $\tau^* : \pi_1(\mathcal{S}^e_\Gamma) \to \pi_1(\mathcal{S}_\Gamma)$. Consider the following isomorphisms

$$\begin{aligned} \tau^* : \pi_1(\mathcal{S}^e_\Gamma, \bullet) &\xrightarrow{\sim} \pi_1(\mathcal{S}_\Gamma, \bullet), \\ \psi_2 : \quad A_\Gamma &\xrightarrow{\sim} \pi_2(\mathcal{S}^e_\Gamma, \bullet), \\ \psi_1 := \tau^* \circ \psi_2 : \quad A_\Gamma &\xrightarrow{\sim} \pi_1(\mathcal{S}_\Gamma, \bullet). \end{aligned}$$

We will use the basepoints in the universal cover of the Salvetti complex from Definition 5.2.8. Now consider a point $v \in \overline{\mathcal{S}^e_\Gamma}$ and $g \in A_\Gamma$, then there is a unique closed path $p_v \in \pi_2(\mathcal{S}^e_\Gamma, \bullet)$ which lifts to a path in $\overline{\mathcal{S}^e_\Gamma}$ from $\bullet^e_0$ to v. Then we define v^g to be the endpoint of to lift of the path $\psi_2(g) \circ p_v \in \pi_1(\mathcal{S}^e_\Gamma, \bullet)$ to the universal cover, we denote this lift by $\overline{\psi_2(g) \circ p_v}$. Similar for a point in $v \in \overline{\mathcal{S}_\Gamma}$.
Step 2: Equivariant proof: Let $v \in \overline{\mathcal{S}^e_\Gamma}$ be a point of the universal cover of the exploded Salvetti complex, let p_v be the path in $\mathcal{S}^e_\Gamma$ that lifts to a path $\overline{p_v}$ starting at $\bullet^e_0$ ending at v in $\overline{\mathcal{S}^e_\Gamma}$. Then $\overline{\tau}(\overline{p_v})$ starts at $\bullet_0$ and ends at $\overline{\tau}(v)$, hence since the endpoints of lifts are determined by the closed path in $\mathcal{S}_\Gamma$ we have endpoint $\overline{p_{\overline{\tau}(v)}} =$ endpoint $\overline{\tau}(\overline{p_v})$. Take $g \in A_\Gamma$ arbitrary, we need to prove that $\overline{\tau}(v^g) = \overline{\tau}(v)^g$.

$$\overline{\tau}(v^g) = \overline{\tau}\left(\text{endpoint } \overline{\psi_2(g) \circ p_v}\right) = \text{endpoint } \overline{\tau}\left(\overline{\psi_2(g) \circ p_v}\right).$$

Because Diagram (5.1) commutes, we have

$$\phi\left(\overline{\tau}\left(\overline{\psi_2(g)\circ p_v}\right)\right) = \tau\left(\phi_e\left(\overline{\psi_2(g)\circ p_v}\right)\right) = \tau\left(\psi_2(g)\circ p_v\right).$$

Hence, the endpoint of the lift of $\tau\left(\psi_2(g)\circ p_v\right)$ coincides with the endpoint of $\overline{\tau}\left(\overline{\psi_2(g)\circ p_v}\right)$.

$$\begin{aligned}\text{endpoint } \overline{\tau}\left(\overline{\psi_2(g)\circ p_v}\right) &= \text{endpoint } \overline{\tau\left(\psi_2(g)\right)\circ\tau\left(p_v\right)}\\ &= \text{endpoint } \overline{\tau^*\left(\psi_2(g)\right)\circ\tau\left(p_v\right)}\\ &= \text{endpoint } \overline{\psi_1(g)\circ p_{\overline{\tau}(v)}} = \overline{\tau}(v)^g.\end{aligned}$$

□

Definition 5.3.9. Since $\mathcal{S}_\Gamma$ and $\mathcal{S}^e_\Gamma$ are spaces with fundamental group A_Γ, there is a natural action of A_Γ on the points of $\overline{\mathcal{S}_\Gamma}$ and $\overline{\mathcal{S}^e_\Gamma}$. This action is defined in the proof of Lemma 5.3.8.

Lemma 5.3.10. *For a vertex $\bullet'$ in $\overline{\mathcal{S}_\Gamma}$ (i.e. $\bullet' \in \phi^{-1}\left((\bullet_v)_{v\in V(\Gamma)}\right)$ see Definition 5.2.6). We have that the inverse image $\overline{\tau}^{-1}(\bullet')^{(0)}$ is isomorphic to the following graph product.*

$$\widetilde{\prod}_{v\in V(\Gamma)}([0,1]_v,\bullet_v) = \bigcup_{\substack{\Delta \subseteq V(\Gamma)\\ clique}}\left(\prod_{v\notin\Delta}\{\bullet_v\}\times\prod_{v\in\Delta}\left([0,1]_v,\bullet_v\right)\right),$$

here $[0,1] = \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$. Moreover, the image of the vertices will map by ϕ_e to the corresponding vertices in $\mathcal{S}^e_\Gamma$.

Proof. In this proof we will regularly apply the commutativity of diagram (5.1). We have $\phi(\bullet') = \bullet \in \mathcal{S}_\Gamma$. By Definition 5.2.4 of τ we have

$$\tau^{-1}(\bullet) = \widetilde{\prod}_{v\in V(\Gamma)}([0,1]_v,\bullet_v) =: L.$$

However this space L is simply connected, hence the connected components of $\phi_e^{-1}(L)$ are all isomorphic to L. Since the Diagram (5.1) commutes, we have that $\overline{\tau}^{-1}(\bullet')$ is contained in such a component call this $\overline{L}_0$. Suppose $\overline{\tau}(\overline{L}_0) \supsetneq \bullet'$, however $\phi\left(\overline{\tau}(\overline{L}_0)\right) = \bullet$ and $\phi^{-1}(\bullet)$ are all disjoint point. Now since $\overline{L}_0$ is connected and $\overline{\tau}$ is continuous $\overline{\tau}(\overline{L}_0)$ needs to be a connected component of the inverse image $\phi^{-1}(\bullet)$. We conclude $\overline{\tau}(\overline{L}_0) = \bullet'$. □

Lemma 5.3.11. *Consider the maps in Definition 5.2.4, then $\overline{\tau} : \mathcal{S}^e_\Gamma \to \mathcal{S}_\Gamma$ maps standard k-flats to standard k-flats.*

Proof. Let $F \subseteq \overline{\mathcal{S}_g^e}$ be a standard flat, i.e., there is a certain clique $\Delta \subseteq \Gamma$ such that F is a connected component of $\phi_e^{-1}\left(\prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}\right)$. Since ϕ_e is a covering map, we have $\phi_e(F) = \prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}$. We also have that

$$\tau\left(\prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}\right) = \prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \bullet_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\} \subseteq \mathcal{S}_\Gamma$$

is a standard torus of $\mathcal{S}_\Gamma$. Since Diagram (5.1) commutes, we have

$$\overline{\tau}(F) \subseteq \phi^{-1}\left(\prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \bullet_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}\right)$$
$$\phi\left(\overline{\tau}(F)\right) = \prod_{v\in\Delta\subseteq V(\Gamma)}(S_v^1, \bullet_v) \times \prod_{v\in V(\Gamma)\setminus\Delta}\{\bullet_v\}.$$

Hence, $\overline{\tau}(F)$ is contained in the inverse image of a k-torus. We are thus left to prove that $\overline{\tau}(F)$ is connected, but this follows from the fact that continuous maps send connected sets to connected sets. □

Lemma 5.3.12. *Let A_Γ be a right-angled Artin group. A subset of vertices $G \subseteq \phi^{-1}(\bullet)$ is the 0-skeleton of a flat F (i.e. $G = F^{(0)}$) in $\overline{\mathcal{S}_\Gamma}$ if and only if G coincides with the orbit[15] of an arbitrary point $x \in G$ by a spherical subgroup $A_\Delta \leq A_\Gamma$, i.e., $x^{A_\Delta} = G(= F^{(0)})$.*

Proof. ⇒: We need to prove that $G = F^{(0)}$ is an orbit. We assume that F is a flat, let $T_\Delta = \phi(F)$ for a certain clique $\Delta \subseteq V(\Gamma)$ (see Remark 5.3.6). Consider $x, y \in F^{(0)}$, then there exists two unique paths $\tilde{x}, \tilde{y} \in \pi(\mathcal{S}_\Gamma, \bullet)$ such that the endpoint of the lift of these paths map to x and y respectively (see Figure 5.3.1).

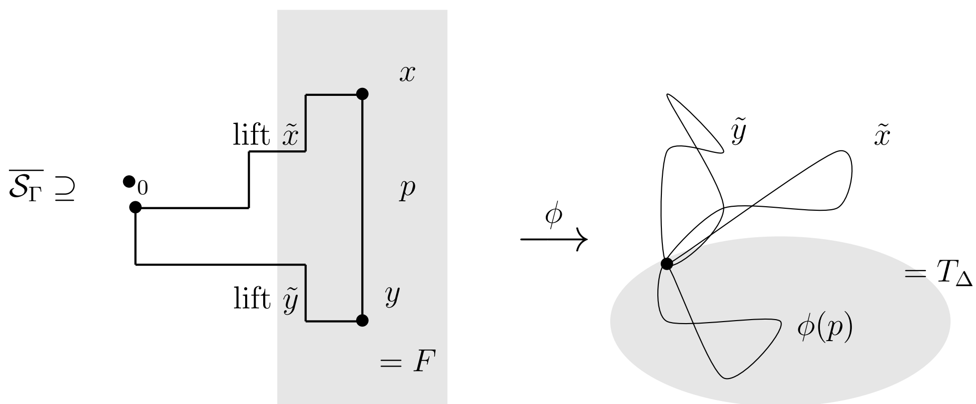


Figure 5.3.1: Our situation in proof of Lemma 5.3.12

[15]of the action Definition 5.3.9

Since F is connected, there is a path $p \subseteq F$ from x to y. Since $x, y \in F^{(0)}$, the path $\phi(p) \in T_\Delta \subseteq \mathcal{S}_\Gamma$ is a path form the base point $\bullet \in \mathcal{S}_\Gamma$ to the base point. Hence, $\phi(p) \in \pi_1(T_\Delta, \bullet) \cong A_\Delta$.

$$x^{\phi(p)} = (\bullet_0)^{\tilde{x}\phi(p)} = (\bullet_0)^{\tilde{y}} = y.$$

The second equality comes from the fact that the endpoints of $p \circ \mathrm{lift}(\tilde{x})$ and $\mathrm{lift}(\tilde{y})$ coincide (being y).
$\Leftarrow$: We need to prove that G is the 0-skeleton of a standard flat. Pick $x \in G$, let F be the connected component containing x of $\phi^{-1}(T_\Delta)$. We prove that $F^{(0)} = G$. Since $G = x^{A_\Delta}$ is given and $x^{A_\Delta} = F^{(0)}$ by "$\Rightarrow$", we have what we wanted. □

The "if and only if" statement in Lemma 5.3.12 is not true for the universal cover of the exploded Salvetti complex. However, the "only if" part is still true.

Lemma 5.3.13. *Let A_Γ a right-angled Artin group. A subset of vertices G is the 0-skeleton of a flat F (i.e. $G = F^{(0)}$) in $\overline{\mathcal{S}^e_\Gamma}$ then G coincides with the orbit of an arbitrary point $x \in G$ by a spherical subgroup $A_\Delta \leq A_\Gamma$, i.e. $x^{A_\Delta} = G (= F^{(0)})$.*

Proof. Similar to the proof of Lemma 5.3.12, with one easy extra argument, since a priori we will not always have that $\phi_e(p) \in \pi_1(\mathcal{S}^e_\Gamma, (\bullet_v)_v)$, since the image $\phi_e(p)$ of a path p between x and y could be a path from vertices of the form $(*_v)_v$ (rather than $(\bullet_v)_v$). However, we can just extend this path in the simply connected part of $\mathcal{S}^e_\Gamma$ to a path starting and ending in $(\bullet_v)_v$. □

One of the strengths of the exploded Salvetti complex, which will be mostly important to define a restriction quotient map to the associated building, is the fact that its flats in the universal cover are disjoint.

Lemma 5.3.14. *Let A_Γ be a right-angled Artin group. The standard tori in $\mathcal{S}^e_\Gamma$ are all disjoint, also the standard flats in $\overline{\mathcal{S}^e_\Gamma}$ are disjoint.*

Proof. For the first part, consider $\Delta \neq \Delta' \subseteq V(\Gamma)$ such that they both span a clique, and let T_Δ and $T_{\Delta'}$ be their respective torus. There exists, without loss of generality $v \in \Delta \setminus \Delta'$, then the coordinate at place v in $T_{\Delta'} \subseteq \mathcal{S}^e_\Gamma$ is always $\bullet_v$, while the coordinates at position v in T_Δ are elements in $(S^1_v, \circ_v) \not\ni \bullet_v$. We are left to prove that different flats in $\overline{\mathcal{S}^e_\Gamma}$ are disjoint.

1. **Case 1:** Consider two flats that correspond to the same standard torus, $F \neq F' \subseteq \phi_e^{-1}(T_\Delta)$, for a clique $\Delta \subseteq V(\Gamma)$. Suppose $x \in F \cap F'$, then $F = x^{A_\Delta} = F'$ by Lemma 5.3.13.

2. **Case 2:** Consider two flats corresponding to different tori such that $v \in$

$\Delta \setminus \Delta'$ and

$$\phi_e(F) = \prod_{v\in\Delta}(S^1_v, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta} \{\bullet_v\},$$
$$\phi_e(F') = \prod_{v\in\Delta'}(S^1_v, \circ_v) \times \prod_{v\in V(\Gamma)\setminus\Delta'} \{\bullet_v\}.$$

Suppose $x \in F \cap F'$, then $(\phi_e(x))_v = \bullet$ in $\phi_e(F')$ while $(\phi_e(x))_v \in (S^1_v, \circ_v)$ in $\phi_e(F)$, which gives us a contradiction. □

In general in the Salvetti complex $\mathcal{S}_\Gamma$ the standard flats do need to be disjoint.

Lemma 5.3.15. *Let A_Γ be a right-angled Artin group. Every vertex $v \in \overline{\mathcal{S}^e_\Gamma}^{(0)}$ is contained in a unique flat.*

Proof. Take a point $v \in \overline{\mathcal{S}^e_\Gamma}^{(0)}$. Then $\phi_e(p) = (*_i)_{i\in V(\Gamma)}$ with $*_i \in \{\circ, \bullet\}$ (Definition 5.2.6). This corresponds to a unique clique in Γ and thus a unique torus T_Δ (where $\Delta := \{i \in V(\Gamma) \mid *_i = \circ\}$). Then $v \in \phi_e^{-1}(T_\Delta)$ such that v is contained in a flat. By Lemma 5.3.14 this flat is unique. □

Lemma 5.3.16. *Let A_Γ be a right-angled Artin group. Write flats($\overline{\mathcal{S}_\Gamma}$) and flats($\overline{\mathcal{S}^e_\Gamma}$) for the set of flats of $\overline{\mathcal{S}_\Gamma}$ and $\overline{\mathcal{S}^e_\Gamma}$ respectively. There are bijections ψ_1 : flats($\overline{\mathcal{S}_\Gamma}$) $\to A_\Gamma\mathcal{S}^f$ (respectively ψ_2 : flats($\overline{\mathcal{S}^e_\Gamma}$) $\to A_\Gamma\mathcal{S}^f$) between the flats of $\overline{\mathcal{S}_\Gamma}$ (respectively $\overline{\mathcal{S}^e_\Gamma}$) and the set $A_\Gamma\mathcal{S}^f$ of spherical cosets of A_Γ that satisfy the following:*

(i) *For a flat $F \in$ flats($\overline{\mathcal{S}_\Gamma}$) for which $\phi(F) = T_\Delta$ we have $\psi_1(F) = gA_\Delta$ for a certain $g \in A_\Gamma$.*

(ii) *For a flat $F \in$ flats($\overline{\mathcal{S}^e_\Gamma}$) for which $\phi_e(F) = T_\Delta$ we have $\psi_2(F) = gA_\Delta$ for a certain $g \in A_\Gamma$.*

(iii) *The following diagram commutes*

$$\begin{array}{ccc} \text{flats}(\overline{\mathcal{S}^e_\Gamma}) & \xrightarrow{\quad\overline{\tau}\quad} & \text{flats}(\overline{\mathcal{S}_\Gamma}) \\ & {\scriptstyle\psi_2}\searrow \quad \swarrow{\scriptstyle\psi_1} & \\ & A_\Gamma\mathcal{S}^f & \end{array} \quad . \tag{5.2}$$

Proof. **(i):** We first prove there is a map ψ_1 that is also is a bijection. Consider the action $A_\Gamma \curvearrowright \overline{\mathcal{S}_\Gamma}$ with base point $\bullet_0 \in \overline{\mathcal{S}_\Gamma}^{(0)}$.

- Flats of rank 0: Consider a flat x of rank 0; this is a connected component of $\phi^{-1}(\bullet)$. By Remark 5.2.9 (ii), we know there is a unique element $a_x \in A_\Gamma$ such that $\bullet_0^{a_x} = x$ (with $\bullet_0$ the base point of $\overline{\mathcal{S}_\Gamma}$). We can now define ψ_1 on the points:
$$\psi_1 : \overline{\mathcal{S}_\Gamma}^{(0)} \to A_\Gamma\mathcal{S}^f : x \mapsto a_x A_\emptyset.$$

- Consider F a flat of rank k for which $x^{A_\Delta} = F$ with $|\Delta| = n$ (which is possible by Lemma 5.3.12 for a $x \in F$). Just as before, there is a unique element $a_x \in A_\Gamma$ such that $\bullet_0^{a_x} = x$. We define:

$$\psi_1 : \text{flats}(\overline{\mathcal{S}_\Gamma}) \to A_\Gamma \mathcal{S}^f : F \mapsto a_x A_\Delta.$$

  We are left to prove that this is well-defined. Consider a different $y \in F^{(0)}$, we want to prove that $a_x A_\Delta = a_y A_\Delta$. However, Lemma 5.3.12 tells us that $x \in F = y^{A_\Delta}$, hence $x \in \bullet_0^{a_y A_\Delta}$, this implies $a_x \in a_y A_\Delta$.

We prove that this is indeed a bijection. Injective: suppose $\psi_1(F) = \psi_1(\tilde{F})$ for two flats, hence, $a_x A_\Delta = a_y A_{\tilde{\Delta}}$ (with $\psi_1(F) = a_x A_\Delta$ and $\psi_1(\tilde{F}) = a_y A_{\tilde{\Delta}}$), this implies $A_\Delta = A_{\tilde{\Delta}}$ and $a_x^{-1} a_y \in A_\Delta$. Such that

$$F = x^{A_\Delta} = x^{a_x^{-1} a_y A_\Delta} = \bullet_0^{a_y A_\Delta} = y^{A_\Delta} = \tilde{F}.$$

Surjective: let aA_Δ be a spherical coset. Consider the flat $(\bullet_0^a)^{A_\Delta}$ by Lemma 5.3.12 the result follows.
(ii): Since $\overline{\tau}$ is equivariant (Lemma 5.3.8), we have that $\overline{\tau}(F^{(0)}) = \overline{\tau}(x^{A_\Delta}) = \overline{\tau}(x)^{A_\Delta} = F'^{(0)}$, this is a flat by Lemma 5.3.12. This is also clearly a bijection.
(iii): We can prove this similarly to how we proved (i), but now sticking to using vertices of $\overline{\mathcal{S}_\Gamma^e}$ in $\phi_e^{-1}\left((*_v)_{v \in V(\Gamma)}\right)$. After doing this one verifies that Diagram 5.2 commutes. However, one can also check that $\psi_2 := \psi_1 \circ \overline{\tau}$ satisfies the asked. □

Example 5.3.17. We continue from Example 5.2.3, Example 5.3.5 and Figure 5.2.2, where the defining graph was $\Gamma := \overset{a}{\bullet}\overset{2}{\text{—}}\overset{b}{\bullet}$. In Figure 2.10.1 (See Construction 2.10.3) one can see the universal cover of the Salvetti complex, which is a $\mathbb{R} \times \mathbb{R}$ grid with 1-skeleton $\mathbb{Z} \times \mathbb{Z}$. The standard flats are the points, horizontal and vertical lines and the $\mathbb{R}^2$ plane itself. We now draw the universal cover of the exploded Salvetti complex

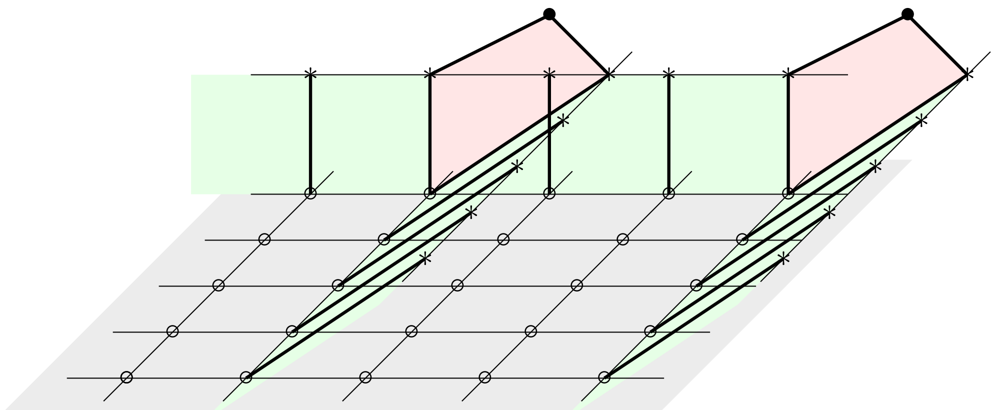


Figure 5.3.2: Universal cover of the exploded Salvetti complex.

In Figure 5.3.2 only a portion of the universal cover is drawn, above every horizontal and vertical line in the gray plane is a green strip. Where two green strips meet there is a red square that would be mapped to the red square in Figure 5.2.2 by the covering map ϕ_e. Hence, for every vertex $\circ$ in the gray grid there is a red square above it. The vertices are labeled $\circ, *$ and $\bullet$ for the following reason, the covering map does the following.

$$\begin{aligned}\phi_e : \overline{\mathcal{S}^e_\Gamma} &\to \mathcal{S}^e_\Gamma : \\ \bullet &\mapsto (\bullet_a, \bullet_b), \\ \circ &\mapsto (\circ_a, \circ_b), \\ * &\mapsto \begin{cases} (\bullet_a, \circ_b) & \text{if } * \text{ is on a horizontal line} \\ (\circ_a, \bullet_b) & \text{if } * \text{ is on a vertical line} \end{cases}.\end{aligned}$$

The standard flats in $\overline{\mathcal{S}^e_\Gamma}$ are all disjoint. The flats are the points with symbol $\bullet$, the horizontal and vertical lines in $\overline{\mathcal{S}^e_\Gamma}$ that exist of $*$ vertices and the gray $\mathbb{R}^2$ plane. For a continuation of this example, see Example 5.5.4.

5.4 Connection between the Salvetti complex, and the set of residues of the associated building

Corollary 5.4.1. *We have that the poset $A_\Gamma \mathcal{S}^f$ is isomorphic to the poset* flats $\left(\overline{\mathcal{S}_\Gamma}\right)$, *and hence*

$$D_\Gamma = \text{Flag}\left(A_\Gamma \mathcal{S}^f\right) \cong \text{Flag}\left(\text{flats}\left(\overline{\mathcal{S}_\Gamma}\right)\right),$$

$$\text{Geom}(\mathcal{B}_{A_\Gamma}) \cong \text{Geom}\left(\text{Flag}\left(A_\Gamma \mathcal{S}^f\right)\right) \cong \text{Geom}\left(\text{Flag}\left(\text{flats}\left(\overline{\mathcal{S}_\Gamma}\right)\right)\right).$$

Proof. We need to prove that the two posets $A_\Gamma \mathcal{S}^f$ and flats$\left(\overline{\mathcal{S}_\Gamma}\right)$ are isomorphic. By Lemma 5.3.16 (i) there is a bijection ψ_1 : flats $\left(\overline{\mathcal{S}_\Gamma}\right) \to A_\Gamma \mathcal{S}^f$. We want to prove that it is an isomorphism between posets, i.e., we need to prove that for cosets gA_Δ and $hA_{\tilde{\Delta}}$ that $hA_{\tilde{\Delta}} \subseteq gA_\Delta$ if and only if they come from flats $\tilde{F}$ and F such that $\tilde{F} \subseteq F$. We first prove the "if" part. Suppose two $\tilde{F} \subseteq F \in$ flats$(\overline{\mathcal{S}_\Gamma})$, pick $x \in \tilde{F}$, hence, by using the construction in the proof of Lemma 5.3.16

$$\begin{aligned}\psi_1 &: \tilde{F} \mapsto a_x A_{\tilde{\Delta}}; \\ \psi_1 &: F \mapsto a_x A_\Delta.\end{aligned}$$

We are left to prove $\tilde{\Delta} \subseteq \Delta$. We have $\phi(\tilde{F}) = T_{\tilde{\Delta}}$ and $\phi(F) = T_\Delta$. Hence, $T_{\tilde{\Delta}} \subseteq T_\Delta$ and $\tilde{\Delta} \subseteq \Delta$. Now the "only if" part. Suppose $hA_{\tilde{\Delta}} \subseteq gA_\Delta$, then we have $hA_{\tilde{\Delta}} \subseteq hA_\Delta = gA_\Delta$. From this it follows that $\tilde{\Delta} \subseteq \Delta$ (also see Remark 1.1.5). Now consider the point $x := \bullet^h_0 \in \overline{\mathcal{S}_\Gamma}$. Then we have two flats $\tilde{F}$ and F

(by Lemma 5.3.12) of the form $\tilde{F}^{(0)} = x^{A_{\tilde{\Delta}}}$ and $F^{(0)} = x^{A_\Delta}$. Since $A_{\tilde{\Delta}} \leq A_\Delta$, we have $\tilde{F} \subseteq F$.

The other isomorphisms follow from Theorem 4.5.3. $\square$

Remark 5.4.2. Corollary 5.4.1 is not true for $\overline{\mathcal{S}^e_\Gamma}$ instead of $\overline{\mathcal{S}_\Gamma}$, since flats in $\overline{\mathcal{S}^e_\Gamma}$ are all disjoint. However, the whole idea behind the exploded Salvetti complex is exactly this. We will use the fact that these flats are disjoint to get an natural map (a restriction quotient map) from $\overline{\mathcal{S}^e_\Gamma}$ to Geom $\left(\mathcal{B}_{A_\Gamma}\right)$ in the next chapter.

5.5 Map between the exploded Salvetti complex and the associated building

In this section we will construct a map between the universal cover of the exploded Salvetti complex and the associated building $\mathcal{B}_{A_\Gamma}$.This map will be a restriction quotient map. In Lemma 5.3.14 we already proved that flats are disjoint. We will now first determine "how" disjoint they are and how they are connected to each other.

In this section we will often use the action of A_Γ on $\overline{\mathcal{S}_\Gamma}$ and $\overline{\mathcal{S}^e_\Gamma}$. This Action was defined in Lemma 5.3.8. This was done by identifying A_Γ with $\pi_1(\mathcal{S}_\Gamma)$, and lifting closed paths in this group to paths in $\overline{\mathcal{S}_\Gamma}$ with basepoint $\bullet_0 \in \phi^{-1}(\bullet)$. Similarly, by identifying A_Γ with $\pi_1(\mathcal{S}^e_\Gamma)$ such that this identification is preserved by $\tau : \mathcal{S}^e_\Gamma \to \mathcal{S}_\Gamma$, and now with basepoint $\bullet^e_0 \in \phi_e^{-1}((\bullet_v)_{v\in V(\Gamma)})$ such that $\overline{\tau}(\bullet^e_0) = \bullet_0$. We also proved that this action equivariant (Lemma 5.3.8).

The following definition can only be defined for the exploded Salvetti complex.

Definition 5.5.1. As discussed in Remark 5.2.5, we know that every point in $\mathcal{S}^e_\Gamma$ is associated to a unique torus. Similarly, for every vertex $x \in \overline{\mathcal{S}^e_\Gamma}$ we write T^x for the *associated torus* T^x of x, that being:

$$T^x := \prod_{*_v=\circ} (S^1_v, \circ_v) \times \prod_{*_v=\bullet} \{\bullet_v\}, \text{where } (*_v)_{v\in V(\Gamma)} := \phi_e(x)$$

i.e., the unique standard torus containing $\phi_e(x)$. By Lemma 5.3.15 we can also associate a unique flat for every point, that we denote by F_x.

It follows directly that the dimension of the torus T^x is precisely the size of the set $\{v \in V(\Gamma) \mid *_v = \circ\}$, with $\phi_e(x) = (*_v)_{v\in V(\Gamma)}$.

Remark 5.5.2. Consider the situation as in Definition 5.5.1.

(i) One can easily determine (similar to the proof of Lemma 5.3.15) that for every vertex $x \in \overline{\mathcal{S}^e_\Gamma}$, we have $\phi_e(F_x) = T^x$.

(ii) Let $V(\Gamma) \supseteq \Delta := \{v \in V(\Gamma) \mid *_v = \circ\}$, where $(*_v)_{v\in V(\Gamma)} := \phi_e(x)$ such that $T_\Delta = T^x$. By Lemma 5.3.12 we have $\overline{\tau}(F^{(0)}_x) = \overline{\tau}(x^{A_\Delta}) = \overline{\tau}(x)^{A_\Delta}$.

Definition 5.5.3. Let $\overline{\mathcal{S}^e_\Gamma}$ bet the universal cover the exploded Salvetti complex. We already defined what vertices are in the universal cover (Definition 5.2.6) We now define the following.

(i) An *edge* $e = \{x, y\}$ in $\overline{\mathcal{S}^e_\Gamma}$ is a 1-cube that is mapped (by the covering map ϕ_e) to a 1-cube in
S^e_Γ (i.e. either to an interval $[0, 1]$ or a circle S^1).

(ii) A *horizontal edge* $e = \{x, y\}$ is an edge in $\overline{\mathcal{S}^e_\Gamma}$ that is a 1-cube that is mapped by ϕ_e to a $[0, 1]$ interval edge in $\mathcal{S}^e_\Gamma$.

(iii) A *vertical edge* is an 1-cube that is mapped to a S^1 circle by ϕ_e in $\mathcal{S}^e_\Gamma$.

Example 5.5.4. We continue where we left of in Example 5.3.17. Consider the case where $\Gamma := \overset{a}{\bullet}\overset{2}{\text{———}}\overset{b}{\bullet}$. The universal cover of the exploded Salvetti complex is drawn in Figure 5.3.2. The vertical edges are the edges between two $\circ$-vertices or two $*$-vertices and the horizontal edges are those between a $\bullet$-vertex and a $*$-vertex or a $\circ$-vertex and a $*$-vertex. We will work further on this example in Example 5.5.4.

Lemma 5.5.5. *An edge $e = \{x, y\}$ in $\overline{\mathcal{S}^e_\Gamma}$ is vertical if and only if x and y are contained in the same flat.*

Proof. $\Rightarrow$: Suppose e is a vertical edge. Hence, it covers a circle $S^1_{v_0}$ for a certain $v_0 \in V(\Gamma)$, thus we can write

$$\phi_e(e) = \prod_{v \in \Delta} \{\circ_v\} \times S^1_{v_0} \times \prod_{\substack{v \notin \Delta \\ v \neq v_0}} \{\bullet_v\},$$

for a certain $\Delta \subseteq V(\Gamma)$ that spans a clique. This is an edge to and from the point $\prod_{v \in \Delta}\{\circ_v\} \times \{\circ_{v_0}\} \times \prod_{\substack{v \notin \Delta \\ v \neq v_0}}\{\bullet_v\}$. Hence, both x and y are contained in $\phi_e^{-1}(T_{\Delta \cup \{v_0\}}) \subseteq \overline{\mathcal{S}^e_\Gamma}$ while they are also connected by the edge $e \subseteq \phi^{-1}\left(T_{\Delta \cup \{v_0\}}\right)$ in $\overline{\mathcal{S}^e_\Gamma}$. Thus, they are in the same flat.

$\Leftarrow$: Suppose e is horizontal while x and y are in the same flat. This would mean that e covers an I_{v_0} interval (for a certain $v_0 \in V(\Gamma)$), i.e., $\phi_e(e)$ is the edge between

$$\prod_{v \in \Delta}\{\circ_v\} \times \prod_{v \notin \Delta}\{\bullet_v\} \text{ and } \prod_{v \in \Delta}\{\circ_v\} \times \{\circ_{v_0}\} \times \prod_{\substack{v \notin \Delta \\ v \neq v_0}}\{\bullet_v\},$$

for a certain $\Delta \subseteq V(\Gamma)$ that spans a clique. This means that $\phi_e(x)$ and $\phi_e(y)$ are not even in the same torus, hence x and y definitely cannot be in the same flat. $\square$

We will now prove that horizontal edges are edges between flats, such that the associated tori of these flats in $\mathcal{S}_\Gamma$ are included in each-other with co-dimension 1.

Lemma 5.5.6. *Consider two points $x, y \in \overline{\mathcal{S}^e_\Gamma}^{(0)}$ with a horizontal edge e between them, such that the endpoints are contained in different flats F_x and F_y, respectively. Then the following is satisfied:*

(i) $\overline{\tau}(x) = \overline{\tau}(y)$,

(ii) $\overline{\tau}(F_x) \subseteq \overline{\tau}(F_y)$ *is a co-dimension*[16] *one subset.*

(iii) $\tau(T^x) \subseteq \tau(T^y)$ *is a co-dimension*[17] *one subtorus.*

Proof. **(i):** This almost directly follows since Diagram (5.1) commutes and τ retracts the $[0, 1]$ part. We will prove this now in detail.

Since e is a horizontal edge we have

$$\phi(e) = \prod_{v \in \Delta} \{\circ_v\} \times I_{v_0} \times \prod_{\substack{v \notin \Delta \\ v \neq v_0}} \{\bullet_v\},$$

hence, the vertices x and y correspond to the tori T_Δ and $T_{\Delta \cup \{v_0\}}$ respectively. Suppose $\overline{\tau}(x) \neq \overline{\tau}(y)$, then $\overline{\tau}(e)$ is an edge between $\overline{\tau}(x)$ and $\overline{\tau}(y)$. Since ϕ is a cover, there is a neighborhood $U \subseteq \overline{\mathcal{S}_\Gamma}$ of $\overline{\tau}(x)$ that does not contain $\overline{\tau}(y)$ and such that $\phi|_U$ is injective (by choosing U small enough). Now $\{\overline{\tau}(x)\} \subsetneq U$, hence, $\{\phi(\overline{\tau}(x))\} \subsetneq \phi(U)$, and also $\{\overline{\tau}(x)\} \subsetneq U \cap \overline{\tau}(e)$ and $\{\phi(\overline{\tau}(x))\} \subsetneq \phi(U) \cap \phi(\overline{\tau}(e))$ since ϕ is injective on U. However

$$\{\phi(\overline{\tau}(x))\} \subsetneq \phi(U) \cap \phi(\overline{\tau}(e)) \subseteq \phi(\overline{\tau}(e)) = \tau(\phi_e(e)) = \tau(I_{v_0}) = \{\bullet\} = \{\phi(\overline{\tau}(x))\},$$

which is a contradiction.
(ii): We have

$$\overline{\tau}(F_x)^{(0)} = \overline{\tau}(x^{A_\Delta}) = \overline{\tau}(x)^{A_\Delta} \subseteq \overline{\tau}(x)^{A_{\Delta \cup v_0}} = \overline{\tau}(y)^{A_{\Delta \cup v_0}} = \overline{\tau}(F_y)^{(0)}.$$

Now since $\mathbb{Z}^{|\Delta|+1} \cong A_\Delta \times \mathbb{Z} \cong A_{\Delta \cup \{v_0\}}$, the statement follows, because the action is free.
(iii): Since $T^x = T_\Delta \subset T_{\Delta \cup \{v_0\}} = T^y$. □

Example 5.5.7. We again continue from Example 5.5.4. The edges between two vertices with the same symbol (i.e. two $\circ$, two $*$ or two $\bullet$ vertices (in Figure 5.3.2)) are vertical edges. Edges between two different symbols are horizontal. They connect two different flats, these two different flats are disjoint and differ by one dimension. The flats in Figure 5.3.2 are the following. The unique one of dimension 2 i.e. the gray plane containing all $\circ$ vertices, this flat corresponds to the complete torus T_Γ or with unique spherical residue of rank two i.e. A_Γ. The flats of rank 1 are the lines which connect $*$ type points (remember we only drew three of them while there is one for every line in the blue plane), these flats

[16] This has meaning, since Remark 5.3.6 tells us that a flat isomorphic is with $\mathbb{R}^n$ for a certain $n \in \mathbb{N}$.
[17] i.e. $T^x = S^{k-1} \subseteq S^k = T^y$ for a certain $k \in \mathbb{N}$

correspond with $T_{\{a\}}$ or $T_{\{b\}}$ and corresponds to cosets of the form $gA_{\{b\}}$ or $gA_{\{a\}}$ with $g \in A_\Gamma$. The flats of rank 0 are the points of the form $\bullet$ (we only drew two in Figure 5.3.2) and correspond to cosets $gA_\emptyset$.

If we collapse the space $\overline{\mathcal{S}^e_\Gamma}$ along these flats (i.e. every flat becomes a point), we will obtain the geometric realization of the right-angled building defined in Theorem 4.5.3. We will now go in more depth. The following was discussed by Huang and Kleiner in [33].

Construction 5.5.8. We will construct a restriction quotient map $q : \overline{\mathcal{S}^e_g} \to \mathrm{Geom}(\mathcal{B}_{A_\Gamma})$ from the universal cover of the exploded Salvetti complex to the geometric realization of the associated building (See Definition 4.5.4). First, we only define it on the vertices of $\overline{\mathcal{S}^e_\Gamma}$:

$$q : \overline{\mathcal{S}^e_\Gamma}^{(0)} \to \mathrm{Geom}(\mathcal{B}_{A_\Gamma})^{(0)} : x \mapsto a_x A_\Delta,$$

where $\Delta \leq V(\Gamma)$ such that $T^x = T_\Delta$ and $a_x \in A_\Gamma$ is the element such that $(\bullet^e_0)^{a_x} = x$.

Lemma 5.5.9. *Consider two vertices $x, y \in \overline{\mathcal{S}^e_\Gamma}^{(0)}$, if there is a horizontal edge e between them, then $q(x) \subsetneq q(y)$ (or $q(x) \supsetneq q(y)$) as sets*[18] *and* rank $q(x) =$ rank $q(y) - 1$ *(or* rank $q(x) - 1 =$ rank $q(y)$ *respectively).*

Proof. We have by Lemma 5.5.6(iii) $\tau(T^x) \subseteq \tau(T^y)$ with co-dimension one. Hence,

$$\begin{aligned} q(x) &= a_{\overline{\tau}(x)} A_{\Delta_x} \text{ with } T^x = T_{\Delta_x}, \\ q(y) &= a_{\overline{\tau}(y)} A_{\Delta_y} \text{ with } T^x = T_{\Delta_y}. \end{aligned}$$

Hence, $A_{\Delta_x} \subseteq A_{\Delta_y}$. By Lemma 5.5.6 (i) the first statement follows. The second statement follows from the fact that, since $\tau(T^x) \subseteq \tau(T^y)$ with co-dimension one, we have $|\Delta_x| + 1 = |\Delta_y|$. ☐

We will now prove, in some sense, the converse.

Lemma 5.5.10. *Consider two cosets $gA_\Delta, hA_{\Delta'} \in A_\Gamma \mathcal{S}^f$ with $gA_\Delta \subseteq hA_{\Delta'}$ and $|\Delta'| = |\Delta| + 1$. Then there exist $x, y \in \overline{\mathcal{S}^e_\Gamma}$ and a horizontal edge e between them such that $q(x) = gA_\Delta$ and $q(y) = hA_{\Delta'}$.*

Proof. Since $gA_\Delta \subseteq hA_{\Delta'}$, one can choose $h = g$, hence, $\Delta \subseteq \Delta' = \Delta \cup \{v_0\}$ for a certain $v_0 \in V(\Gamma)$. Consider the vertex $\bullet^g_0 \in \overline{\mathcal{S}_\Gamma}$. By Lemma 5.3.10 we know that

$$\overline{\tau}^{-1}(\bullet^g_0) \cong \widetilde{\prod}_{v \in V(\Gamma)} \left([0,1]_v, \bullet_v\right) = \bigcup_{\substack{\Delta \subseteq V(\Gamma) \\ \text{clique}}} \left(\prod_{v \notin \Delta} \{\bullet_v\} \times \prod_{v \in \Delta} \left([0,1]_v, \bullet_v\right) \right),$$

[18] Since $q(x)$ is a vertex in *Geom*$(\mathcal{B}_{A_\Gamma})$ and thus correspond to a coset in $A_\Gamma \mathcal{S}^f$.

here $[0,1] =$ $\bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$. Choose the edge from $x := (\bullet_v)_{v \notin \Delta'} \times (\circ_v)_{v \in \Delta'} \in \overline{\tau}^{-1}(\bullet_0^g)$ and $y := (\bullet_v)_{v \notin \Delta} \times (\circ_v)_{v \in \Delta} \in \overline{\tau}^{-1}(\bullet_0^g)$. This edge is horizontal by construction. While $x \in \phi_e^{-1}(T_{\Delta'})$ and $y \in \phi_e^{-1}(T_\Delta)$. □

Lemma 5.5.11. *For two points $x, y \in \overline{\mathcal{S}_\Gamma^e}$ with $\operatorname{rank}(T^x) + 1 = \operatorname{rank}(T^y)$ and $\overline{\tau}(x) = \overline{\tau}(y)$, there is a horizontal edge $e \subseteq \overline{\mathcal{S}_\Gamma^e}$ such that $e = \{x, y\}$.*

Proof. Since $\overline{\tau}(x) = \overline{\tau}(y) =: \bullet' \in \overline{\mathcal{S}_\Gamma}^{(0)}$, we have by Lemma 5.3.10

$$x, y \in \widetilde{\prod}_{v \in V(\Gamma)} \left([0,1]_v, \bullet_v\right) \cong \overline{\tau}^{-1}(\bullet') \subseteq \overline{\mathcal{S}_\Gamma^e}.$$

This subspace only exist of horizontal edges. It is easy to check that they are here just one horizontal edge apart since w.l.o.g. $T^x = T_\Delta \subseteq T_{\Delta \cup \{v\}} = T^y$. □

Lemma 5.5.12. *Consider two edges e_1, e_2 in $\overline{\mathcal{S}_\Gamma^e}$ such that they are opposite in a 2-cube, then we have*

- *if e_1 is horizontal then e_2 is horizontal;*
- *if e_1 is vertical then e_2 is vertical.*

Proof. Suppose e_1 and e_2 are opposite in a 2-cube C. Then C is the product of e_1 and e_1' i.e. $C = [0,1]^2 = e_1 \times e_1'$. Now we look at

$$\phi_e(e_1 \times e_1') = \phi_e(e_1) \times \phi_e(e_1') \subseteq \mathcal{S}_\Gamma^e = \widetilde{\prod}_{v \in V(\Gamma)} \left(L_v, \bullet_v\right).$$

However, the cubes of the exploded Salvetti complex are always of the form either $S^1 \times S^1$, $[0,1] \times [0,1]$ or $S^1 \times [0,1]$. Either way the opposite edges are always of the same form (both a circle or both a $[0,1]$ interval). Hence, the image of e_2 is of the same form as e_1, which is what we wanted to prove. □

Construction 5.5.13 ([33, Section 5]). We continue where we left off in Construction 5.5.8. Since vertices of vertical edges are mapped by q to the same vertices and vertices of horizontal edges are mapped by q to two different points which are cosets gA_Δ and $gA_{\Delta'}$ (where $|\Delta| + 1 = |\Delta'|$), hence, their image is connected by an edge is $\operatorname{Geom}(\mathcal{B}_{A_\Gamma})$. It makes sense to define the following.

$$q : \overline{\mathcal{S}_\Gamma^e}^{(1)} \to \operatorname{Geom}(\mathcal{B}_{A_\Gamma})^{(1)} :$$
$$e = \{x, y\} \mapsto \begin{cases} \text{the edge } \{a_x A_\Delta, a_x A_{\Delta'}\} & \text{if } e \text{ is horizontal,} \\ \text{the point } a_x A_\Delta & \text{if } e \text{ is vertical.} \end{cases}$$

We will now extend this to a restriction quotient map. A horizontal cube in $\overline{\mathcal{S}_\Gamma^e}$ is a cube where each of its edges is a horizontal edge, a vertical cube is a cube such that each of its edges is a vertical edge. Every cube in $\overline{\mathcal{S}_\Gamma^e}$ is a product of horizontal cube and a vertical cube. The image of the vertex set of a horizontal cube spans a cube in $\mathrm{Geom}(\mathcal{B}_{A_\Gamma})$ and the image of the vertex set of a vertical cube is a vertex in $\mathrm{Geom}(\mathcal{B}_{A_\Gamma})$. We thus can extend q to a cubical map $q : \overline{\mathcal{S}^e_\Gamma} \to \mathrm{Geom}(\mathcal{B})$. We can construct p also as a restriction quotient map as follows. Let $\hat{\mathcal{H}}$ be the set of all hyperplanes of $\overline{\mathcal{S}^e_\Gamma}$, let $\hat{\mathcal{K}}$ be the set of hyperplanes dual to a horizontal edges in $\overline{\mathcal{S}^e_\Gamma}$. Then one can verify that we can identify q as the following restriction quotient map

$$q = q : \overline{\mathcal{S}^e_\Gamma} \cong X(\hat{\mathcal{H}}) \to X(\hat{\mathcal{K}}) \cong \mathrm{Geom}(\mathcal{B}_{A_\Gamma}).$$

Example 5.5.14. We will construct this restriction quotient map for the case where $\Gamma :=$ $\overset{a}{\bullet}\!\overset{2}{\text{———}}\!\overset{b}{\bullet}$. In Figure 5.3.2 we drew a part of the universal cover of the exploded Salvetti complex, which was also discussed in Example 5.3.5 and the edges that are horizontal and vertical where discussed in Example 5.3.17. We draw a portion of the universal cover again, additionally we draw the horizontal edges thick (these are the edges between different flats).

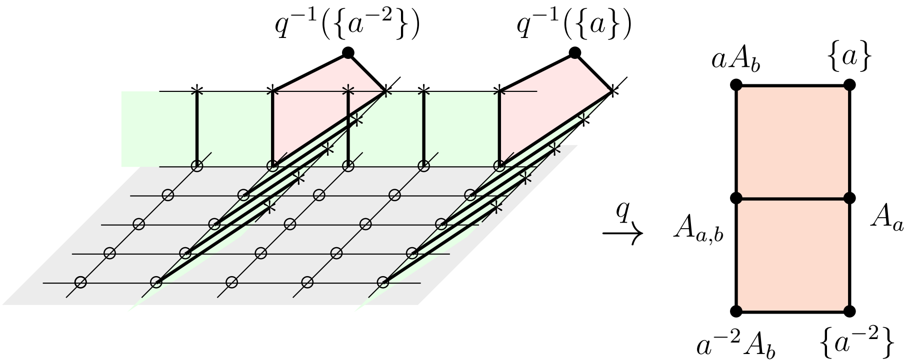


Figure 5.5.1: The restriction quotient map

For every edge of the form $e = \{\{a\}, aA_b\}$ in $\mathrm{Geom}(\mathcal{B}_{A_\Gamma})$ between a rank 0 residue (i.e. a chamber) and a rank 1 residue, there is one edge in $\overline{\mathcal{S}^e_\Gamma}$ that maps to this edge. For every edge of the form $e = \{aA_b, A_{\{a,b\}}\}$, there are $\mathbb{Z}$ many edges in $\overline{\mathcal{S}^e_\Gamma}$ that will map to this edge. The inverse image of $A_{\{a,b\}}$ is $\mathbb{Z} \times \mathbb{Z}$. If we want to see this map as a restriction quotient map we will choose the following hyperplanes $\hat{\mathcal{K}} \subseteq \hat{\mathcal{H}}$ ($\hat{\mathcal{H}}$ the set of all hyperplanes). The two hyperplanes in each red $[0,1] \times [0,1]$ cube we keep in $\hat{\mathcal{K}}$. For every green strip we keep the hyperplane that is the line that goes through the whole length of the strip dual to the edge of the form ∘——∗ . Having this we can verify that

$$q = q : \overline{\mathcal{S}^e_\Gamma} \cong X(\hat{\mathcal{H}}) \to X(\hat{\mathcal{K}}) \cong \mathrm{Geom}(\mathcal{B}_{A_\Gamma}).$$

We discuss some properties of this map.

Lemma 5.5.15 ([33, p566])**.** *This restriction quotient map is an A_Γ-equivariant map.*

In Example 5.5.14 we discussed the cardinality of the inverse image of vertices in $\mathrm{Geom}(\mathcal{B})$. We will now prove the following general result.

Lemma 5.5.16. *For every vertex $gA_\Delta \in \mathrm{Geom}(\mathcal{B}_{A_\Gamma})^{(0)}$, we have $q^{-1}(gA_\Delta)^{(0)} \cong \mathbb{Z}^{|\Delta|}$.*

Proof. The set $q^{-1}(gA_\Delta)$ contains all the vertices that are contained in a unique flat F of type T_Δ, thus by Lemma 5.3.13 $F^{(0)} = x^{A_\Delta}$ for $x \in F$. It now follows since this action is free and $A_\Delta \cong \mathbb{Z}^{|\Delta|}$. □

Lemma 5.5.17. *Consider a cube $\sigma \subseteq \mathrm{Geom}(\mathcal{B}_{A_\Gamma})$ and take $x \in \mathrm{int}(\sigma)$. Let r be the minimal rank of all vertices of $\sigma^{(0)}$. We have $q^{-1}(x) \cong \mathbb{Z}^r$.*

Proof. See [33, Lemma 5.1]. □

Remark 5.5.18 (Alternative construction Exploded Salvetti complex)**.** The exploded Salvetti complex can also be constructed differently. This was done by Kleiner and Bestvina in [9] for right-angled Artin groups of dimension 2 (i.e. Γ does not contain cliques of size ≥ 3). In this paper they also construct a map to the so-called "flat space", this space coincides with the modified Deligne complex and geometric realization of the associated building, and this map is precisely the restriction quotient map from Construction 5.5.13.

The section we just discussed we did a lot to prove that we have a restriction quotient map to $\mathrm{Geom}\left(\mathcal{B}_{A_\Gamma}\right)$. This will be useful in next section where we will prove that a restriction quotient map to $\mathrm{Geom}\left(\mathcal{B}_{A_\Gamma}\right)$ satisfying Lemma 5.5.17, will be equivalent to a blow-up data. This equivalence we will then use in Section 5.7. Unfortunately this thesis primarily gives a detailed description of these objects rather than giving the useful properties they have. For example, the reader of this thesis will probably have little clue that theorems like Theorem 6.2.2 as the results in Section 6.5, Section 6.6 and Section 6.7 use the theory we discussed just now.

5.6 Fiber functor

In this section if we write $\mathcal{B}_{A_\Gamma}$ we mean the right-angled buildings associated to the right-angled Artin group A_Γ (Definition 4.5.4). If we just write $\mathcal{B}$ we mean an arbitrary right-angled building of a fixed type Γ.

Definition 5.6.1 ([33, Section 4.2])**.** Suppose $q : X \to Y$ is a restriction quotient map between two CAT(0) cube complexes. Denote $\mathrm{Face}(Y)$ to be the poset of faces of Y viewed as a category (i.e. the objects are the faces and the morphisms are the inclusions). Denote **CCC** to be the category of nonempty CAT(0) cube complexes and whose morphisms are the convex cubical embeddings.
The following contravariant functor Ψ_q is called the *fiber functor of the restriction*

quotient map q.

$$\begin{aligned} \Psi_q : \mathrm{Face}(Y) \to \mathbf{CCC} :& \\ \sigma &\mapsto q^{-1}(y) \text{ where } y \text{ is an interior point of } \sigma, \\ i_{\sigma_1,\sigma_2} &\mapsto \left(q^{-1}(y_2) \hookrightarrow q^{-1}(y_1)\right) \text{ which exists by Lemma 2.3.17,} \end{aligned}$$

for subcubes $\sigma_1 \subseteq \sigma_2$. This functor is independent of the choice of $y_i \in \mathrm{int}(\sigma_i)$ ([33, Lemma 4.5 (2)]).

Definition 5.6.2. A contravariant functor $\Psi : \mathrm{Face}(Y) \to \mathbf{CCC}$ is 1-*determined* if it satisfies that for every cube $\sigma \in \mathrm{Face}(Y)$ and every vertex $v \in \sigma^{(0)}$ we have

$$\mathrm{Im}\left(\Psi(\sigma) \hookrightarrow \Psi(v)\right) = \bigcap_{v \subsetneq e \subseteq \sigma^{(1)}} \mathrm{Im}\left(\Psi_q(e) \hookrightarrow \Psi_q(v)\right).$$

Lemma 5.6.3 ([33, Lemma 4.14])**.** *A fiber functor Ψ_q of a restriction quotient map q is 1-determined.*

Theorem 5.6.4 ([33, Theorem 4.15])**.** *Let Y be a* CAT(0) *cube complex and $\Psi : \mathrm{Face}(Y) \to \mathbf{CCC}$ be a 1-determined contravariant functor. Then there exist a* CAT(0) *cube complex X and a restriction quotient map $q : X \to Y$ such that the associated fiber functor (as in Definition 5.6.1) $\Psi_q : \mathrm{Face}(Y) \to \mathbf{CCC}$ is equivalent by a natural transformation to Ψ.*

Sketch of proof. We only give the construction of the cube complex X. We start with the following cube complex

$$\bigsqcup_{\sigma \in \mathrm{Face}(Y)} \left(\sigma \times \Psi(\sigma)\right).$$

For every inclusion $\sigma_1 \subseteq \sigma_2$ we glue $\sigma_1 \times \Psi(\sigma_2) (\subseteq \sigma_2 \times \Psi(\sigma_2))$ to $\sigma_1 \times \Psi(\sigma_1)$ using the following map

$$\sigma_1 \times \Psi(\sigma_2) \xrightarrow{id_{\sigma_1} \times \Psi(i_{\sigma_1,\sigma_2})} \sigma_1 \times \Psi(\sigma_1).$$

This map is an injection. The resulting space is then X. We then define the restriction quotient map as

$$q : \sigma \times \Psi(\cdot) \mapsto \sigma.$$

□

We will now instead of constructing a contravariant functor from a restriction quotient map, we will construct such a functor from the blow-up data of a building.

Definition 5.6.5 ([33, Page 571])**.** Let $\mathcal{B}$ be an arbitrary right-angled building of type Γ, with a blow-up data $\mathcal{H} := \{h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{\mathrm{rank}(\mathcal{R})}\}$ (see Definition 4.4.1).

We define the *fiber functor associated with the blow-up date* $\mathcal{H}$ as the following functor.

$$\begin{aligned}\Psi_{\mathcal{H}} : \text{Face}\big(\text{geom}(\mathcal{B})\big) &\to \mathbf{CCC} : \\ \sigma &\mapsto \mathbb{R}^{\text{rank}(\mathcal{R}_\sigma)}, \\ i_{\sigma_1,\sigma_2} &\mapsto h_{\mathcal{R}_{\sigma_2},\mathcal{R}_{\sigma_1}},\end{aligned}$$

where $\mathcal{R}_\sigma$ is the residue associated to the unique vertex $v \in \sigma^{(0)}$ of minimal rank[19]. The map $h_{\mathcal{R}_{\sigma_2}\mathcal{R}_{\sigma_1}}$ is defined in Lemma 4.4.3 (which we can use since $\mathcal{R}_{\sigma_2} \subseteq \mathcal{R}_{\sigma_1}$ if $\sigma_1 \subseteq \sigma_2$). The cube complex $\mathbb{R}^n$ has a cubical structure where we identify its 0-skeleton with $\mathbb{Z}^n$.

We give the commuting Diagram form Lemma 4.4.3 again as a reminder.

$$\begin{array}{ccc} \mathcal{R}' & \xhookrightarrow{\quad i \quad} & \mathcal{R} \\ \Big\downarrow h_{\mathcal{R}'} & & \Big\downarrow h_{\mathcal{R}} \\ \mathbb{Z}^{\text{rank}(\mathcal{R}')} & \xrightarrow{h_{\mathcal{R}',\mathcal{R}}} & \mathbb{Z}^{\text{rank}(\mathcal{R})} \end{array} \tag{5.3}$$

As one would expect we now get a Lemma similar to lemma 5.6.3.

Lemma 5.6.6 ([33, Lemma 5.9 and Lemma 5.10])**.** *We have that $\Psi_{\mathcal{H}}$ of Definition 5.6.5 is a 1-determined contravariant functor.*

Lemma 5.6.7. *Let $q_{\mathcal{H}} : X \to \text{Geom}(\mathcal{B})$ be the restriction quotient map constructed in Theorem 5.6.4 from a fiber functor $\Psi_{\mathcal{H}} : \text{Face}\big(\text{Geom}(\mathcal{B})\big) \to \mathbf{CCC}$ associated with blow data $\mathcal{H}$. Then $q_{\mathcal{H}}$ satisfies*[20] *Lemma 5.5.17.*

Proof. Consider a cube $\sigma \subseteq \text{Geom}(\mathcal{B})$ and $x \in \text{int}(\sigma)$. We have

$$q^{-1}(\sigma) = \sigma \times \Psi(\sigma) = \sigma \times \mathbb{R}^{\text{rank}(\mathcal{R}_\sigma)},$$

where the second equality comes from Definition 5.6.5. Hence, we have

$$q^{-1}(w) = \{w\} \times \Psi(\sigma) = \{w\} \times \mathbb{R}^{\text{rank}(\mathcal{R}_\sigma)} \cong \mathbb{R}^{\text{rank}(\mathcal{R}_\sigma)}. \qquad \square$$

Definition 5.6.8 ([33, Defintion 5.3])**.** Let $q : Y \to \text{Geom}(\mathcal{B})$ be a restriction quotient map from a CAT(0) cube complex to Geom($\mathcal{B}$), with $\mathcal{B}$ an arbitrary right-angled building of type Γ. Suppose q satisfies Lemma 5.5.17. Let Ψ_q be the associated fiber functor $\Psi_q : \text{Face}\big(\text{geom}(\mathcal{B})\big) \to \mathbf{CCC}$ (form Definition 5.6.1). Then the 1-*data* associated of q is collection of maps $\{f_{\mathcal{R}} \mid \mathcal{R} \text{ rank 1 residue}\}$ where

$$\begin{aligned} f_{\mathcal{R}} : \mathcal{R} &\to \Psi_q(v_{\mathcal{R}}) : \\ c &\mapsto \Psi_q(i_{v_{\mathcal{R}},e_c})\left(\Psi_q\left(e_c\right)\right),\end{aligned}$$

[19]This exists from the definition of how these cubes are defined Construction 4.3.4.

[20]i.e. $q_{\mathcal{H}}(x) \cong \mathbb{Z}^r$, where r is the minimal rank of all the vertices of $\sigma^{(0)}$ and where σ is the cube in Geom($\mathcal{B}$) such that $x \in \text{int}(\sigma)$

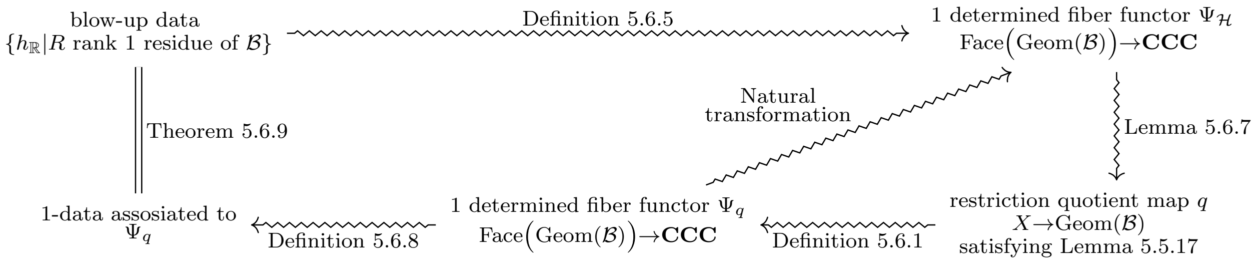
where $v_{\mathcal{R}}$ is the vertex in Geom($\mathcal{B}$) that correspond with the residue $\mathcal{R} =: \mathcal{R}_v$, e_c is the 1-cube between c and $v_{\mathcal{R}}$ in Geom($\mathcal{B}$). The map $i_{v_{\mathcal{R}},e_c} : v_{\mathcal{R}} \hookrightarrow e_c$ is the inclusion of the vertex $v_{\mathcal{R}}$ in the edge e_c. Since the vertices in Geom($\mathcal{B}$) are the spherical residues of $\mathcal{B}$, we will also write $\mathcal{R}_v$, to indicate that the residue $\mathcal{R}$ corresponds to the vertex v. By definition (see Definition 5.6.1) $f_{\mathcal{R}_v}$ coincides with

$$f_{\mathcal{R}_v} : \mathcal{R}_v \to q^{-1}(y) \cong \mathbb{R}^{\operatorname{rank}(\mathcal{R}_v)} : \\ c \mapsto \operatorname{im}\left(q^{-1}(y_c) \hookrightarrow q^{-1}(y)\right) \text{ This embedding exists by Lemma 2.3.17,}$$

with y_c an interior point of e_c.

Theorem 5.6.9 ([33, Theorem 5.11])**.** *Let* $\mathcal{H} := \{h_{\mathcal{R}} \mid \mathcal{R}$ rank 1 residue$\}$ *a blow-up data (Definition 4.4.1) and* $\Psi_{\mathcal{H}}$ *the associated fiber functor (Definition 5.6.5). Let* q *be the restriction quotient map constructed in Theorem 5.6.4. Then the 1-data of* q *is the blow-up data we started with.*

Remark 5.6.10. We will give a summary of the previous theorems and definitions



This diagram commutes in the sense that if we start in one place and apply the theorem/definition in this order one would get maps/sets equivalent with the one we start with.

5.7 Fiber equivalences applying on Exploded Salvetti complex

Theorem 5.7.1 ([33, Theorem 5.19])**.** *Let* $\mathcal{B}$ *be an arbitrary right-angled building of type* Γ *with a blow-up data*

$$\mathcal{H} = \{h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{\operatorname{rank}(\mathcal{R})} \mid \mathcal{R} \text{ a spherical residue}\}$$

such that each map $h_{\mathcal{R}}$ *is bijective. Then* X *constructed from this blow-up data (Theorem 5.6.4) is isomorphic to the universal cover of the exploded Salvetti complex* $\overline{\mathcal{S}^e_\Gamma}$.

Sketch of proof. We construct a covering map $p : X \to \mathcal{S}^e_\Gamma$. Having this we prove that $X \cong \overline{\mathcal{S}^e_\Gamma}$ by looking at the lift $\tilde{p} : X \to \overline{\mathcal{S}^e_\Gamma}$. □

As corollary of this theorem one also can prove that there is only one right-angled building of type Γ and thickness $\vec{q} := (\mathbb{Z}_i)_{i \in V(\Gamma)}$, which we already stated in Theorem 4.3.9.

Corollary 5.7.2 ([33, Corollary 5.21])**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two right-angled buildings of type Γ. Suppose that the rank 1 residues of both buildings are countably infinitely big. Then they are isomorphic buildings.*

Sketch of proof. Consider a bijective blow-up for each of these buildings[21], call them respectively $\mathcal{H}_1$ and $\mathcal{H}_2$. Construct the associated restriction quotient maps

$$q_1 : Y_1 \to \mathrm{Geom}(\mathcal{B}_1);$$
$$q_2 : Y_2 \to \mathrm{Geom}(\mathcal{B}_2).$$

By Theorem 5.7.1, there are covering maps

$$p_1 : Y_1 \to \mathcal{S}^e_\Gamma;$$
$$p_2 : Y_2 \to \mathcal{S}^e_\Gamma.$$

They then left to isomorphism

$$\tilde{p_1} : Y_1 \tilde{\to} \overline{\mathcal{S}^e_\Gamma};$$
$$\tilde{p_2} : Y_2 \tilde{\to} \overline{\mathcal{S}^e_\Gamma}.$$

By[22] Lemma 5.1.11 these maps descend to isomorphisms

$$\tilde{p_1} : \mathrm{geom}(\mathcal{B}_1) \tilde{\to} \mathrm{geom}(\mathcal{B}_{A_\Gamma});$$
$$\tilde{p_2} : \mathrm{geom}(\mathcal{B}_1) \tilde{\to} \mathrm{geom}(\mathcal{B}_{A_\Gamma}).$$

□

[21] It is easy to prove (as exercise) that this always exists given that each rank 1 residue is cardinality $|\mathbb{Z}|$.

[22] One first verifies that these maps $\tilde{p}_i$ are cubical isomorphism.

6 Quasi-isometries

There are many interesting quasi-isometric right-angled Artin groups. One of them is the set of right-angled Artin groups for which the defining graph Γ is a tree of diameter at least 3 [6], as well as a generalization [5]. The interesting papers of Huang [32], [33] and [31], classify some classes rigid (i.e. they are quasi-isometric if and only if they are isomorphic) right-angled Artin groups.

This section is not only interesting for the results themselves but also to see application on why this Salvetti complex is useful, especially in Section 6.9.

We start this section with the Definition of quasi-isomeric spaces, as well as some theorems for right-angled Artin groups. After this section we will delve in the constructions of Huang en Kleiner, there we will use complexes we have discussed like the Salvetti complex, as well as a new complex being the extension complex.

6.1 Definitions

Definition 6.1.1 (Quasi-isometric)**.** Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f : M_1 \to M_2$ be a map satisfying the following:

$$(\exists A, B \in \mathbb{N})(\forall x, y \in M_1)\left(\frac{1}{A}d_1(x,y) - B \leq d_2\big(f(x), f(y)\big) \leq Ad_1(x,y) + B\right) \tag{6.1}$$

and

$$(\exists C \in \mathbb{N})(\forall z \in M_2)(\exists x \in M_1)\Big(d_2\big(z, f(x)\big) \leq C\Big). \tag{6.2}$$

Then we call f a *quasi-isometry* and the spaces M_1 and M_2 *quasi-isometric*. If only equation (6.1) is satisfied we call f and *quasi-isometric embedding*.

Definition 6.1.2. Two finitely generated groups are *quasi-isometric* if their Cayleygraphs (with the word metric) are quasi-isometric[23].

[23]It can be shown that this is independent of the choice of the finite generating set of these groups.

Theorem 6.1.3 ([26, theorem 1])**.** *Two right-angled Artin groups are isomorphic if and only if their defining graphs are isomorphic.*

Lemma 6.1.4 ([8])**.** *Let* Γ *be a graph and* Λ *a subgraph. Then the injection* $A_\Lambda \hookrightarrow A_\Gamma$ *induces a quasi-isometric embedding between* $\overline{\mathcal{S}_\Lambda} \to \overline{\mathcal{S}_\Gamma}$*.*

Sketch of proof. Clearly there is an isometric embedding between $\mathcal{S}_\Lambda \hookrightarrow \mathcal{S}_\Gamma$. By lifting this to an isometric embedding between $\overline{\mathcal{S}_\Lambda} \hookrightarrow \overline{\mathcal{S}_\Gamma}$, we get what we wanted. □

Remark 6.1.5. There are plenty of examples of non-isomorphic right-angled Artin groups that are quasi-isometric. For example free groups of finite rank ≥ 2 are all quasi-isometric. In Section 6.9 we will prove that for every right-angle Artin group A_Γ for which the defining graph has diameter at least 3, there are non-isomorphic Artin groups that are quasi-isometric to A_Γ. However, there will be classes of right-angle Artin group that are rigid i.e. they are quasi-isometric to each other if and only if they are isomorphic.

6.2 Role of exploded Salvetti complex

Definition 6.2.1 (Geometric action [27])**.** Let G be a group that acts on a metric space (X, d_X). This is a *geometric action* if the following is satisfied:

(i) The action is *isometric* (i.e. for every element $g \in G$ and every pair of points $x_1, x_2 \in X$ we have $d_X(x_1, x_2) = d_X(x_1^g, x_2^g)$).

(ii) The action is *properly discontinuous* (i.e. for all $g \in G$ and $x \in X$ there exist a neighborhood of x being $U_{gx} \subseteq X$ such that $U_{gx}^g \cap U_{gx} = \emptyset$).

(iii) The action is *cocompact* (i.e. X/G is compact).

The following theorem follows from discussion in Section 5.7, after doing some more work see [33, Section 5 & 6].

Theorem 6.2.2 ([33, Theorem 6.5])**.** *Let* $\mathcal{B}$ *be any right-angled building of type* Γ*. Let there be a restriction quotient map* $q : Y \to Geom(\mathcal{B})$ *that satisfies Lemma 5.5.17. If a group* G *acts geometrically on* Y*, then is* G *quasi-isometric to* A_Γ*.*

6.3 Extension complex

In this section we will see what the Extension complex is of a right-angled Artin group. It will turn out these complexes will contain a lot of information of the possible quasi-isometric class of a right-angled Artin group.

Definition 6.3.1. Consider a right-angled Artin group A_Γ and let $\overline{\mathcal{S}_\Gamma}$ be the universal cover of the Salvetti complex. The *extension complex* $\mathcal{P}_\Gamma$ of A_Γ is the flag complex that consists of the following:

(i) The set of vertices $\mathcal{P}^{(0)}$ is the set of parallel[24] classes of geodesics in $\overline{\mathcal{S}_\Gamma}$.

(ii) Two vertices $v, w \in \mathcal{P}^{(0)}$ are connected by a line in $\mathcal{P}^{(1)}$ if there exist geodesics $g_1 \in v$ and $g_2 \in w$ such that $\langle g_a, g_2 \rangle$ spans a 2-flat in $\overline{\mathcal{S}_\Gamma}$.

(iii) A set of k vertices form a k-simplex in $\mathcal{P}^{(k)}$ if and only if these points form a clique in $\mathcal{P}^{(1)}$.

Definition 6.3.2 ([34, Definition 1.2])**.** Let A_Γ be a right-angled Artin group. The *extension graph* $\mathcal{P}(\Gamma)^{(1)}$ of Γ is the graph with vertex set $\{g^{-1}vg \mid g \in A_\Gamma, v \in V(\Gamma)\}$ and two vertices are adjacent if and only if they commute.

As the notation of the extension graph already does imply, we do have the following.

Property 6.3.3 ([32, Lemma 4.2])**.** *The extension graph of a right-angled Artin group is isomorphic with the* 1*-skeleton of its extension complex.*

Lemma 6.3.4 ([33, Definition 3.5])**.** *There is a one-to-one correspondence between k-simplices in* $\mathcal{P}(\Gamma)$ *and parallel classes of standard* $(k+1)$ *flats in* $\overline{\mathcal{S}_\Gamma}$*.*

Proof. Exercise. □

We will give some examples of extension complexes, these complexes are almost always infinite. This example should give the reader some intuition in these complexes. We also recommend the reader to draw sketches of the universal covers of their Salvetti complexes to see the parallel classes yourself, and try to deduce from this what the extension complex is.

Example 6.3.5. (i) Let Γ be the complete graph on n vertices, hence a finite type right-angled Artin group. Then the extension complex $\mathcal{P}(\Gamma)$ is also finite. It is the n-simplex.

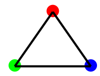
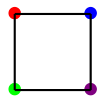
(ii) Let Γ be a set of $n \geq 2$ points without any edges i.e. our Artin group is just the free group of rank n. The extension complex $\mathcal{P}(\Gamma)$ is a space of countably infinite disjoint points (0 simplices) and no higher dimensional simplices.

(iii) Let Γ be a star with at least 3 vertices (or equivalent a star that is not a complete graph). Then the extension complex is a star consisting of countably infinite many leafs.

(iv) Let Γ be a tree of diameter at least 3. Then the extension complex is an infinite graph where a vertex is a leaf or a vertex with countably infinite many neighbors from which there are also countably infinite many neighbors that are not leafs (and maybe also countably infinite many neighbors that are leafs (this depends on Γ)).

[24]two geodesics in $\overline{\mathcal{S}_\Gamma}$ are parallel if they contained in a $\mathbb{R}^2$ subspace and disjoint. The parallel classes are the equivalence classes of the equivalence relation generated by being parallel.

(v) Let Γ be a square, then the extension complex consists of countably infinite many squares such that every vertex is contained in infinitely many squares.

In the figures of following examples we color the vertices (and hence the generators of A_Γ) of the defining graphs. We color a vertex in the extension complex in the color of the generator if this parallel set maps to this generator by the covering map $\phi : \overline{\mathcal{S}_\Gamma} \to \mathcal{S}_\Gamma$. Consider the following defining graphs $\Gamma_1 :=$  , $\Gamma_2 :=$, $\Gamma_3 :=$ and $\Gamma_4 :=$  . The extension complexes of these graphs are drawn in Figure 6.3.1.

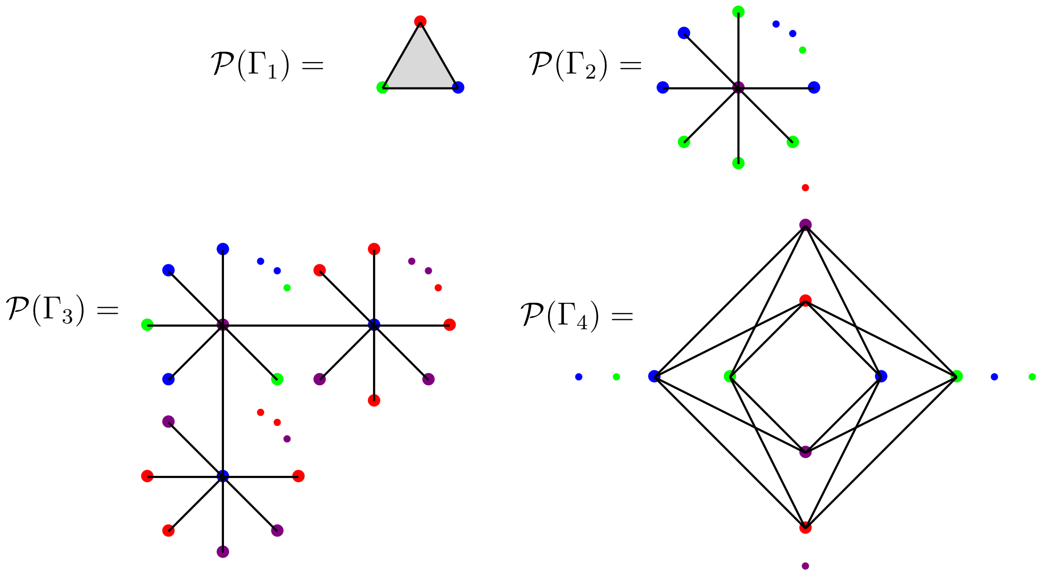
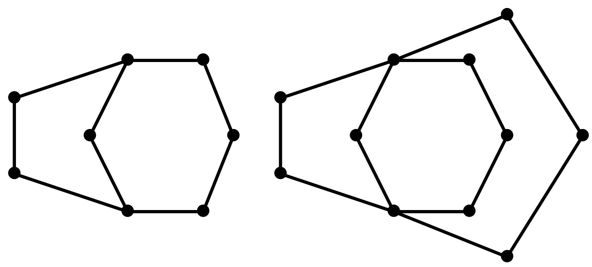


Figure 6.3.1: Extension complexes of the graphs $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

Remark 6.3.6. It is known that all free groups of rank ≥ 2 are quasi-isometric. Clearly by Example 6.3.5 (ii) are the extension complexes of these Artin groups also isomorphic. The extension complexes of Example 6.3.5 (iii) are also all isomorphic to each other, and these groups $\mathbb{Z} \times F_{n-1}$ (F_{n-1} the free group of rank $n-1$, where $n-1$ is the amount of leafs) are all quasi-isometric. We could ask the question:

> **Question.** Is A_{Γ_1} quasi-isometric to A_{Γ_2} if and only if the extension complexes are isomorphic $\mathcal{P}(\Gamma_1) \cong \mathcal{P}(\Gamma_2)$.

In Theorem 6.6.3 en Theorem 6.7.3 we give partially positive answer to this question. However, in general this is not the case. For the only if part we will see in Section 6.8 that the Artin groups defined by $P_5 :=$ and $P_4 :=$ are quasi-isometric while there extension graphs are not (you can prove this by seeing that $\mathcal{P}(P_5)$ contains vertices of distance 2 to a leaf, while such vertices do not exist in $\mathcal{P}(P_4)$). The "if" part is also not true in generals ([31, Example 6.38]). The Artin group of the following two graphs are not quasi-isometric while their extension complexes are isomorphic.



Before going over the quasi-isometric implications these structure will have, we finish this section with a theorem that already shows the importance of the extension complex .

Theorem 6.3.7 ([34, Theorem 1.3]). *Let A_Γ an d A_Λ be two right-angled Artin groups. If Λ is a subgraph of, $\mathcal{P}(\Gamma)^{(1)}$, then A_Λ is isomorphic to a subgroup of A_Γ.*

6.4 Outer automorphism group of a right-angled Artin group

One of the classes of rigid right-angled Artin groups, is the class with finite outer automorphism group. At first this may sound random, however as the following properties will show, having a finite outer automorphism group is equivalent with a concrete structure of the defining graph.

Property 6.4.1 ([47, 32, page 3]). *The outer automorphism group* $\mathrm{Out}(A_\Gamma)$ *of a right-angle Artin group, is generated by the following elements*

(i) For $v \in V(\Gamma)$, $\phi_v : m \mapsto \begin{cases} v^{-1} & \text{if } v = m; \\ m & \text{if } m \in V(\Gamma) \setminus \{v\}. \end{cases}$

(ii) Graph automorphism of Γ.

(iii) *Transvections*: for $v, w \in V(\Gamma)$ with $lk(w) \subseteq St(v)$ we have

$$\psi_{v,w} : m \mapsto \begin{cases} mv & \text{if } m = w; \\ m & \text{if } m \in V(\Gamma) \setminus \{w\}. \end{cases}$$

(iv) *Partial conjugation*: for $v \in V(\Gamma)$ with $\Gamma \setminus St(v)$ disconnected, let $C \subseteq \Gamma$ be a connected component we have

$$\theta_{v,C} : m \mapsto \begin{cases} v^{-1}mv & \text{if } m \in V(C); \\ m & \text{if } m \in V(\Gamma) \setminus V(C). \end{cases}$$

Corollary 6.4.2. *The outer automorphism group* $\mathrm{Out}(A_\Gamma)$ *is finite if and only if there are no vertices* $v, w \in V(\Gamma)$ *such that* $lk(w) \subseteq St(v)$ *or such that* $\Gamma \setminus St(v)$ *is disconnected.*

Sketch of proof. Follows from Property 6.4.1 and the fact that only the Transvections (iii) and Partial conjugations (iv) have infinite order, and (i) and (ii) commute. □

6.5 Atomic right-angled Artin groups

In this subsection we will discuss a class of rigid right-angled Artin groups. These will be at most 2-dimensional.

Definition 6.5.1 ([9, Definition 1.5])**.** A connected graph Γ is *atomic* if it satisfies the following:

(i) every vertex $v \in V(\Gamma)$ has degree at least 2;

(ii) there are no cycles of length smaller than 5;

(iii) for every vertex $v \in V(\Gamma)$, the graph $\Gamma \setminus St(v)$ is connected.

A right-angled Artin group is called *Atomic* if its defining graph is atomic.

Theorem 6.5.2 ([9, corollary 1.7])**.** *Atomic right-angled Artin groups are quasi-isometric if and only if they are isomorphic.*

Sketch of proof. Suppose there is a quasi-isometry, they prove that there is a special kind of isomorphism between the associated flat spaces[25] ([9, Theorem 8.10]), and from this they prove that there is an isomorphism between the defining graphs. □

6.6 Finite Out(A_Γ) case

In this section we will see that right-angled Artin groups that have finite outer automorphism groups are quasi-isometric if and only if they are isomorphic. The extension complex is here crucial, as it will follow from following lemma.

Lemma 6.6.1 ([32, Lemma 4.6])**.** *Let* A_Γ *and* A_Λ *be two right-angled Artin groups with* $\mathrm{Out}(A_\Gamma)$ *and* $\mathrm{Out}(A_\Lambda)$ *finite. Every quasi-isometry* $\chi : \overline{\mathcal{S}_\Gamma} \to \overline{\mathcal{S}_\Lambda}$ *induces a simplicial isomorphism* $\chi^* : \mathcal{P}(\Gamma) \xrightarrow{\sim} \mathcal{P}(\Lambda)$*. If only* $\mathrm{Out}(A_\Gamma)$ *is finite then* χ^* *is still a simplicial embedding.*

Lemma 6.6.2 ([32, Corollary 4.16])**.** *If two right-angled Artin groups* A_Γ *and* A_Λ *with finite outer have isomorphic extension complexes, then they are isomorphic.*

[25]these spaces correspond with the associated building, see Remark 5.5.18

Sketch of proof. Suppose we have an isomorphism between extension complexes $\chi^* : \mathcal{P}(\Gamma) \to \mathcal{P}(\tilde{\Gamma})$. The idea is now to reconstruct a map $\chi' : A_\Gamma \to A_{\tilde{\Gamma}}$ from χ^*. We will define this map as follows. If $x \in \overline{\mathcal{S}_\Gamma}^{(0)}$, denote $\{F_i\}_i$ the set of maximal standard flats in $\overline{\mathcal{S}_\Gamma}$ containing x. Every maximal flat F_i correspond with a maximal simplex c_i in $\mathcal{P}(\Gamma)$ also see Lemma 6.3.4. By applying χ^* on these simplices, we get a set of simplices $\{\tilde{c_i}\}_i := \{\chi^*(c_i)\}_i$ and then picking the corresponding maximal flats in $\overline{\mathcal{S}_{\tilde{\Gamma}}}$, we get a set of $\{\tilde{F_i}\}_i$ of flats in $\overline{\mathcal{S}_{\tilde{\Gamma}}}$. We then define

$$\chi' : A_\Gamma \to A_{\tilde{\Gamma}} : x \mapsto \cap_i \tilde{F_i}. \tag{6.3}$$

However, in general $\cap_i \tilde{F_i}$ is not just one point, it could be empty or bigger. It turns out in the finite $\mathrm{Out}(A_\Gamma)$ case we always have that $\cap_i \tilde{F_i} \neq \emptyset$ after doing a some work this is a singleton. Having this map one can try to find an isomorphism between the defining graphs □

Theorem 6.6.3 ([32, Theorem 1.1])**.** *Two right-angled Artin groups A_Γ and A_Λ with finite outer automorphism group are isomorphic if and only if they are quasi-isometric if and only if their extension complexes are isomorphic.*

Proof. (1) ⇒ (2): Trivial. (2) ⇒ (3): By Lemma 6.6.1. (3) ⇒ (1): By Lemma 6.6.2. □

If only one group has finite outer automorphism group, Lemma 6.6.1 still gives us the following result.

Theorem 6.6.4 ([32, Theorem 1.2])**.** *Let A_Γ and A_Λ be two right-angled Artin group such that A_Γ as finite outer automorphism group, then the following are equivalent.*

(i) *A_Γ and A_Δ are quasi-isomorphic;*

(ii) *A_Λ is isomorphic to a finite index subgroup of A_Γ;*

(iii) *The extension complexes $\mathcal{P}(\Gamma)$ are $\mathcal{P}(\Lambda)$ isomorphic.*

6.7 Infinite $\mathrm{Out}(A_\Gamma)$ case

Definition 6.7.1 ([31, Defnition 1.1])**.** A graph Γ is *weak of type I* if it satisfies the following:

(i) there exist no vertex $v \in V(\Gamma)$ such that $\Gamma \setminus St(v)$ is disconnected;

(ii) there do not exist two vertices $v, w \in \Gamma$ such that $d(v, w) = 1$ and $\Gamma = St(v) \cup St(w)$.

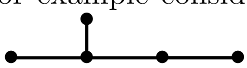
We call a right-angled Artin group of *weak type I* if its defining graph is weak type I.

The point of this section is to prove that week type I Artin groups are rigid. This is done very similar to finite outer automorphism case. We now generalize Lemma 6.6.1.

Theorem 6.7.2 ([31, Theorem 1.11])**.** *Let A_Γ and A_Δ be two right-angled Artin groups. Suppose that* $\mathrm{Out}(A_\Gamma)$ *and* $\mathrm{Out}(A_\Delta)$ *do not contain any nonadjacent transvections. Then every quasi-isometry $\chi : \overline{\mathcal{S}_\Gamma} \to \overline{\mathcal{S}_\Delta}$ induces a simplicial isomorphism $\chi^* : \mathcal{P}(\Gamma) \tilde{\to} \mathcal{P}(\Delta)$.*

Theorem 6.7.3 ([31, Theorem 3.31])**.** *Suppose A_Γ and $A_{\tilde{\Gamma}}$ are two right-angled Artin groups of weak type I. then they are quasi-isometric if and only if they are isomorphic if and only if their extension complexes are isomorphic.*

The proof is similar as the finite out case.

Remark 6.7.4. Suppose we have two right-angled Artin groups such that their extension complexes are isomorphic. We would like to prove that they are quasi-isometric. The proof of Lemma 6.6.2 gives us an idea how to do this; by defining our map as in equation (6.3). However, in general $\cap_i \tilde{F}_i$. could be empty. For example consider the following two graphs $\Gamma_1 =$ and $\Gamma_2 =$ . Their Artin groups are quasi-isometric (see Section 6.8) and their extension complexes are also isomorphic. However, for arbitrary isomorphism $\chi : \mathcal{P}(\Gamma_1) \tilde{\to} \mathcal{P}(\Gamma_2)$. The map χ^* from equation (6.3) will most likely yield to a situation where $\cap \tilde{F} = \emptyset$ (you can easily find such an isomorphism by drawing their extension complexes (also see Example 6.3.5 (iv)) and then looking at corresponding flats in $\overline{\mathcal{S}_{\Gamma_1}}$ and $\overline{\mathcal{S}_{\Gamma_2}}$, and seeing that we will get disjoint flats).

There are more classes (see the papers by Huang, in particular [31]) of rigid right-angled Artin groups then discussed here.

6.8 Defining graph a tree of diameter ≥ 3

Theorem 6.8.1. *All right-angled Artin groups for which their defining graph is a tree of diameter at least* 3 *are quasi-isometric.*

Theorem 6.8.1 was first proven by Behrstock and Neumann in [6] as a corollary of a more general theorem of quasi-isometric classification of fundamental groups of graph manifolds. A different proof was given by Margolis in [37] using JSJ tree of cylinders decomposition of a right-angled Artin Group. Neumann also proved a more general result in [5] for Artin groups that have bisimilar defining graphs.

It is also interesting to notice that many of these quasi-isometric groups do not have isomorphic extension complexes. Nonetheless, they do look like each other, since if we remove all leafs of all these extension complexes then they are isomorphic (they will al be infinite trees where every vertex has countably infinite many neighbors).

We will not prove Theorem 6.8.1; instead, we will prove another theorem (Theorem 6.9.4) in next section. This theorem will give a quasi-isometry for some trees of diameter at least 3. Moreover, we will also show (Corollary 6.9.10) that the theorem that we will prove is insufficient to find quasi-isometries for all trees of diameter at least 3.

6.9 k-double of a defining graph

Let A_Γ be a right-angled Artin group. In this section we construct right-angled Artin groups that are finite index subgroups of A_Γ. Hence, they are also quasi-isometric. The main theorem we will prove gives us a process that can construct from a defining graph Γ a bigger defining graph Γ' such that the Artin groups associated to them are quasi-isometric. This process can be applied to any graph (not just to trees). It will show that for every right-angled Artin group with diameter of defining graph at least 3 there are non-isomorphic quasi-isometric right-angled Artin groups. These constructed quasi-isometric Artin groups will never be of weak type I or have finite outer automorphism group.

Definition 6.9.1. Let Γ be a defining graph of an Artin group. Take L a subgraph of $V(\Gamma)$, and let k be a natural number ≥ 2. Denote $\Gamma\{L, k\}$ for the *k-double of* Γ *along* L this is the graph obtained by gluing k copies of $\Gamma \setminus L$ to L.

Example 6.9.2. In Figure 6.9.1 an example is shown for a graph Γ and its 2-double along the star of $v \in V(\Gamma)$.

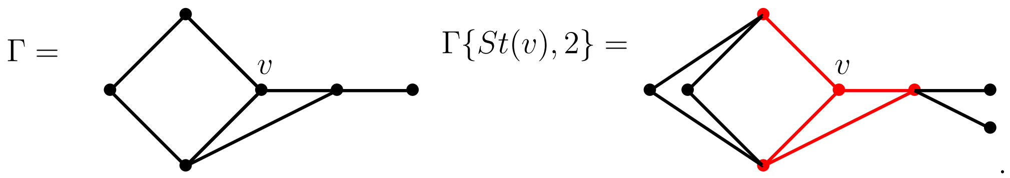


Figure 6.9.1: Graph Γ and its 2- double along $St(v)$.

The point is now to prove that after doing a k-double the resulting right-angled Artin group is quasi-isometric to the Artin group we started with. We will prove this by finding a quasi-isometry between the universal covers of the Salvetti complexes, for this we will need the following well-known lemma in topology.

Lemma 6.9.3 (The general lifting lemma [41, Lemma 79.1]). *Let $p : E \to B$ be a covering map. Consider two points $e_0 \in E, b_0 \in B$ such that $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ and the following diagram commutes*[26]

[26]by definition of lifting [41, Chapter 9 Section 54]

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow{\tilde{f}} & (E, e_0) \\ & \searrow^{f} \quad \swarrow_{p} & \\ & (B, b_0) & \end{array} \tag{6.4}$$

if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)),$$

here $f_* : \pi_1(Y, y_0) \to \pi_1(E, e_0) : q \mapsto f \circ q$. *Furthermore, if such a lift exists, it is unique.*

The proof of this lemma is not necessarily interesting for our theory of right-angled Artin groups. That said, the construction of this map $\tilde{f}$ will be crucial to proof Theorem 6.9.4 (ii) & (iii). Hence, we give this construction.

Sketsh of proof. We want to construct a map $\tilde{f} : Y \to E$ such that diagram (6.4) commutes. Pick $y \in Y$ arbitrary, since Y is path connected we find a path $\alpha : [0,1] \to Y$ such that $\alpha(0) = y_0$ and $\alpha(1) = y$. Now consider the path $\alpha' := f \circ \alpha : [0,1] \to B$. We now lift this to a path $\tilde{\alpha'} : [0,1] \to E$ starting at e_0 in the cover E. We then define

$$\tilde{f}(y) := \tilde{\alpha'}(1).$$

□

We now have sufficient amount of information to prove that right-angled Artin groups of k-doubles are quasi-isometric to the group we started with. The proof itself is not difficult or does not need crazy ideas. It just follows after you notice that the Artin group of the k-double is isomorphic to the kernel of the morphism α in next theorem.

Theorem 6.9.4. *Let* A_Γ *be a right-angled Artin group, and* $v_0 \in V(\Gamma)$ *a vertex. Consider the morphism* α *that we define on the generators as follows:*

$$\alpha : A_\Gamma \to \mathbb{Z}/k\mathbb{Z} : s \mapsto \begin{cases} 1 & \text{if } s = v_0; \\ 0 & \text{if } s \in V(\Gamma) \setminus \{v_0\}. \end{cases}$$

Then the following holds:

(i) *The kernel of* α *is isomorphic to the fundamental group* $\pi_1\left(\overset{k}{\mathcal{S}_\Gamma}\right)$, *where* $\overset{k}{\mathcal{S}_\Gamma}$ *is a k-fold cover of* $\mathcal{S}_\Gamma$, *such that we have* $\pi_1\left(\overset{k}{\mathcal{S}_\Gamma}\right) \cong A_{\Gamma\{St(v),k\}}$.

(ii) *There is a deformation retract*[27] $\chi : \overset{k}{\mathcal{S}_\Gamma} \to \mathcal{S}_{\Gamma\{St(v),k\}}$ *that induces a quasi-isometry* $\bar{\chi} : \overline{\mathcal{S}_\Gamma} \to \overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$.

[27] i.e. a map which is a continuous deformation, it implies that the fundamental groups coincide

(iii) *The extension complex* $\mathcal{P}(\Gamma)$ *of* Γ *is isomorphic to the extension complex* $\mathcal{P}(\Gamma\{St(v),k\})$ *of* $\Gamma\{St(v),k\}$. *Moreover, this isomorphism is induced by* χ.

Hence, A_Γ *and* $A_{\Gamma\{st(v),k\}}$ *are quasi-isometric.*

We will prove Theorem 6.9.4 via constructing a covering map of the Salvetti complex, the idea of the proof of Theorem 6.9.4 (i) comes from [9, Section 11]. However, one can still prove that $\ker(\alpha) \cong A_{\Gamma\{st(v),k\}}$ without using the Salvetti complex but rather in a combinatorial way. This is done by Bell in [7].

Proof. Before proving the real stuff, we will need to do two parts of preparation.
Part 1: Construction of $\overset{k}{\mathcal{S}_\Gamma}$. Since the kernel of α is a subgroup of $\pi_1(\mathcal{S}_\Gamma)$, we have that there is a covering space $\overset{k}{\mathcal{S}_\Gamma}$ of $\mathcal{S}_\Gamma$ that is a k-fold cover (i.e. for all points $x \in \mathcal{S}_\Gamma$ the size of the inverse image of the covering map of x is k). Such that $\pi_1(\overset{k}{\mathcal{S}_\Gamma}) \cong \ker(\alpha)$. This covering space is precisely $\overline{\mathcal{S}_\Gamma}/\ker(\alpha)$. We construct $\overset{k}{\mathcal{S}_\Gamma}$ explicitly. Let ψ be the covering map between the universal cover $\overline{\mathcal{S}_\Gamma}$ and $\overset{k}{\mathcal{S}_\Gamma}$, i.e.

$$\psi : \overline{\mathcal{S}_\Gamma} \to \overset{k}{\mathcal{S}_\Gamma} : \\ \bullet \mapsto \bullet^{\ker(\alpha)}.$$

The set of vertices in the universal cover $\overline{\mathcal{S}_\Gamma}$ of $\mathcal{S}_\Gamma$ (i.e. the set $\phi_1^{-1}(\bullet)$) is in bijection with the set of elements in A_Γ. The kernel of ϕ correspond to the set $\bullet_0^{\ker(\phi)} \subseteq \overline{\mathcal{S}_\Gamma}^{(0)}$ (action from Definition 5.3.9 and $\bullet_0$ was the basepoint such that the lifting of every paths in $\pi_1(\mathcal{S}_\Gamma)$ starts at $\bullet_0$), here $\bullet_0$ is the lift of the trivial path in $\pi_1(\mathcal{S}_\Gamma)$ and $\bullet_0^g$ correspond with the endpoint of the lift of the path $g \in A_\Gamma = \pi_1(\mathcal{S}_\Gamma)$. We know $v_0 \in A_\Gamma \setminus \ker(\alpha)$. Such that $v_0 \notin \ker(\alpha)$ and $v_0 \notin \bullet^{\ker(\alpha)}$ (where we identify A_Γ with $\overline{\mathcal{S}_\Gamma}^{(0)}$), while $v_0^k \in \bullet^{\ker(\alpha)}$. The endpoints of every other path corresponding to a generator $w \in V(\Gamma) \setminus \{v_0\} \subseteq A_\Gamma = \pi_1(\mathcal{S}_\Gamma)$ is also in $\bullet_0^{\ker(\alpha)}$, because $w \in \ker(\alpha)$. Hence $\psi(v_0^k)$ is a closed loop from $\bullet^{\ker(\alpha)}$ to $\bullet^{\ker(\alpha)}$, and so is $\psi(w)$ for all $w \in V(\Gamma) \setminus \{v_0\}$.

Consider an arbitrary cliques C in $Lk(v_0)$. Hence, $C \cup \{v_0\}$ forms a clique, construct the stretched torus $k \cdot S^1 \times \prod_C S^1$ (this is just a $|C|+1$ torus where the circle corresponding to v_0 has length k) This stretched torus is going once around the torus in the w-direction for all $w \in C$ (since after having $w \in Lk(v_0)$ one time it is already $\mathbb{1}$) and going k many times around the torus in the v_0-direction (also see Figure 6.9.2 as an example when $v_0 \sim w$ in Γ and $k = 2$).

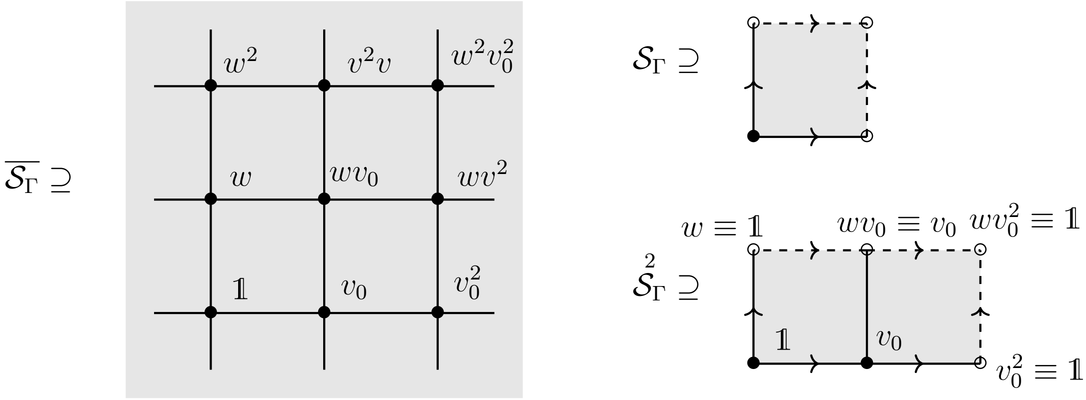


Figure 6.9.2: Showing a part of the universal cover (if $v_0 \sim w$), the 2-fold cover and the Salvetti complex containing $v_0 \in \pi_1(\mathcal{S}_\Gamma)$ or its lifts

Doing this for every clique in $Lk(v_0)$ and if we glue these stretched tori together on the $k \cdot S^1$ part, we get a space that we call L (isomorphic to $(k \cdot S^1) \times \tilde{\prod}_{Lk(v_0)} S^1$). Now consider $\Gamma_0 := \Gamma \setminus \{v_0\}$ (i.e. the graph Γ without the vertex v and all the edges containing v). For all $w \in V(\Gamma \setminus \{v_0\})$ we have $w \in \ker(\alpha)$. We construct the Salvetti complex $\mathcal{S}_{\Gamma_0}$ of Γ_0. We then attach $\mathcal{S}_{\Gamma_0}$ k-many times to L along the edges that correspond to vertices in $V(St(v_0)) \cap V(\Gamma_0)$. See Figure 6.9.3 as an example.

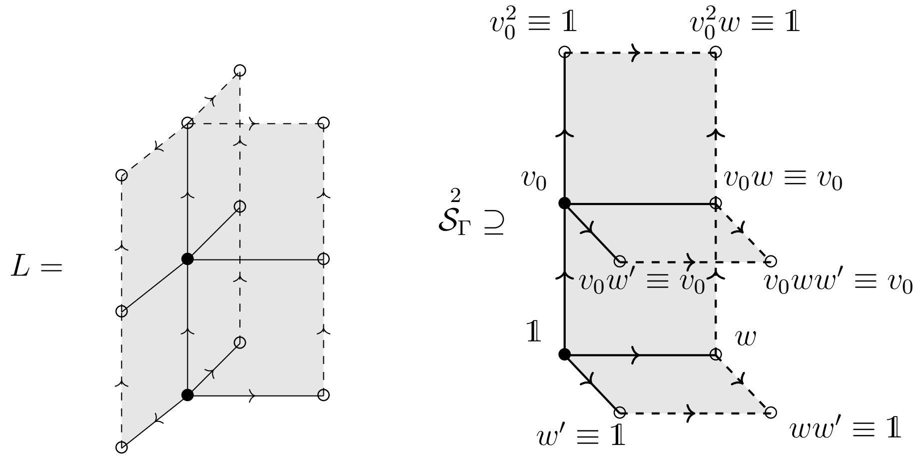


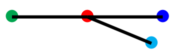
Figure 6.9.3: Showing a part of the 2-fold cover if $v_0 \sim w \sim w' \not\sim v_0$.

To summarize. The k-fold cover can be constructed as follows. One starts with the subgraph that is the star containing v_0 and its neighbors we get a fan of stretched tori in the v direction. On these tori we glue k times the Salvetti complex of $\Gamma \setminus \{v_0\}$.

Part 2: Construction of χ. By Lemma 6.9.3 there is a map $\psi : \overline{\mathcal{S}_\Gamma} \to \overset{k}{\mathcal{S}}_\Gamma = \overline{\mathcal{S}_\Gamma}/\ker(\phi) : x \mapsto x^{\ker(\alpha)}$, this is a continuous map that is also a covering. We now construct a map

$$\chi : \overset{k}{\mathcal{S}}_\Gamma \to \mathcal{S}_{\Gamma\{St(v,k)\}},$$

that does the following it collapses every stretched torus to a normal torus, i.e. $(k \cdot S^1) \times S^1 \to S^1 \times S^1$, this is a homotopy equivalence since a path in $\overset{k}{\mathcal{S}}_\Gamma$ consists of going over circles S^1 or over the circle of length k being $k \cdot S^1$ (the map χ just makes the length of our circle of length k again length 1).

In Figure 6.9.4 we draw the situation for $k = 2$, and we double around the star around •, where it happens to be the case that • is adjacent to • in Γ and also is • adjacent to •, while • is not adjacent to •, i.e. $\subseteq \Gamma$. Finely we get  $\subseteq \Gamma\{St(\bullet), 2\}$.

(i): Since χ is a deformation retract the fundamental groups; $\pi_1(\overset{k}{\mathcal{S}}_\Gamma)$ and $\pi_1(\mathcal{S}_{\Gamma\{St(v),k\}})$ are equal (see [41, Chapter 9 Section 58 & Theorem 58.7.]). Hence, we have that $A_{\Gamma\{St(v),k\}}$} is isomorphic to a subgroup of A_Γ with index k being the kernel of α.

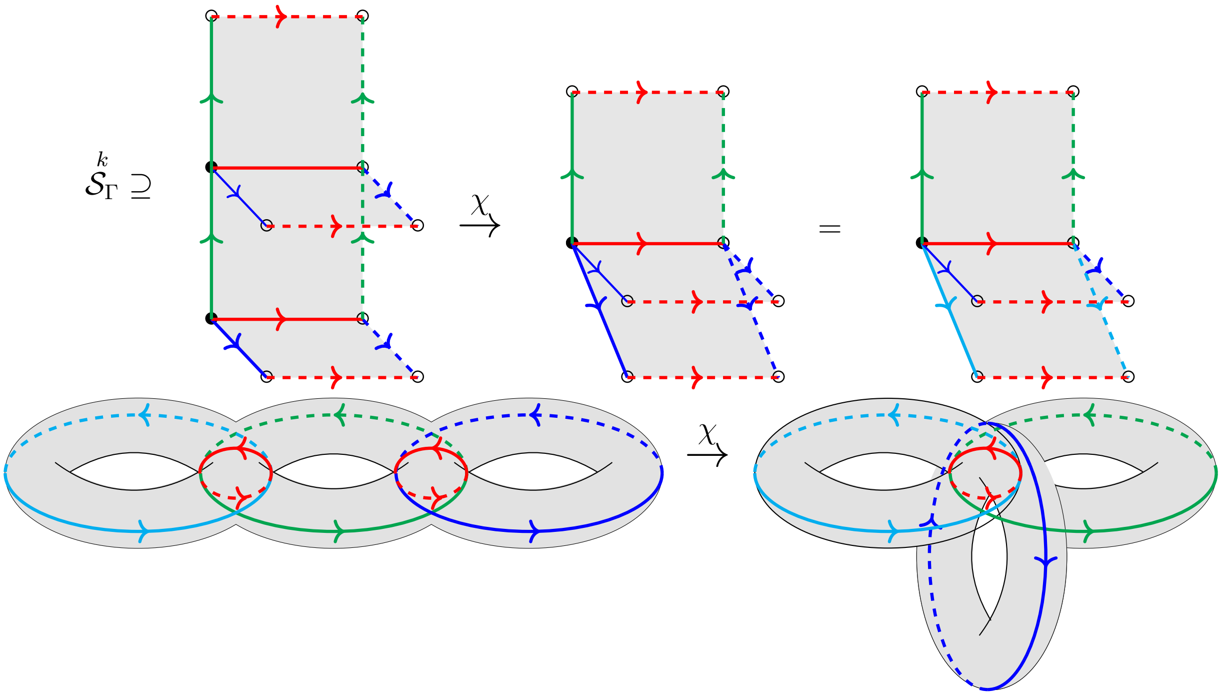
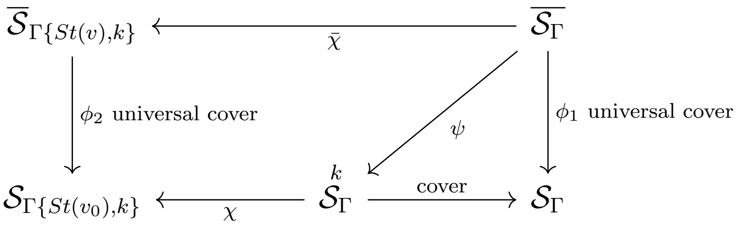


Figure 6.9.4: Collapsing length k circle to length 1 circle.

(ii): The composition $\chi \circ \psi$ is a continuous map. Since $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$ is a covering of $\mathcal{S}_{\Gamma\{St(v,k)\}}$, we can apply the general lifting lemma 6.9.3 to find $\overline{\chi}$. Hence, we have the following commuting diagram.



To prove that $\overline{\chi} : \overline{\mathcal{S}_\Gamma} \to \overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$ is a quasi-isometry, we will need to use the construction used in the proof of Lemma 6.9.3. We claim that $\overline{\chi}$ is a quasi-isometry with the following parameters

$$\frac{1}{k} d_{A_\Gamma}(x, y) - k \leq d_{A_{\Gamma\{St(v),k\}}}(\overline{\chi}(x), \overline{\chi}(y)) \leq k d_{A_\Gamma}(x, y) + k.$$

First notice that it is sufficient to prove this on the 1-skeleton in $\overline{\mathcal{S}_\Gamma}$, and it is sufficient to prove that

$$\frac{1}{k} d_{A_\Gamma}(x, \bullet_0) - k \leq d_{A_{\Gamma\{St(v),k\}}}(\overline{\chi}(x), \bullet_0') \leq k d_{A_\Gamma}(x, \bullet_0) + k,$$

where $\bullet_0' := \overline{\chi}(\bullet_0)$ the basepoint in $\mathcal{S}_{\Gamma\{St(v),k\}}$ (such that for every path in $\mathcal{S}_{\Gamma\{St(v),k\}}$ its lifting starts at $\bullet_0'$). Consider a vertex g in $\overline{\mathcal{S}_\Gamma}^{(0)} = A_\Gamma$. There is a (minimal) path $p := (e_1, e_2, \ldots, e_l)$ between $\bullet_0$ and g consisting of edges (the image by ϕ_1 of this path in $\mathcal{S}_\Gamma$ coincides with the element $g \in A_\Gamma = \pi_1(\mathcal{S}_\Gamma)$). The image of every edge e_i by ϕ_1 is a generator of $\pi_1(\mathcal{S}_\Gamma)$. Now we look at the image of these edges by $\chi \circ \psi$. If $\phi_1(e_i)$ is not equal to the edge/generator in $\pi_1(\mathcal{S}_\Gamma)$ corresponding to $v_0 \in V(\Gamma)$ or to an element in $Lk(v_0)$. Then $\chi\left(\psi(e_i)\right)$ correspond to one of the edges of the form $e_{i_1}', e_{i_2}', \ldots e_{i_k}'$ (the k copies of e_i) in $\mathcal{S}_{\Gamma\{St(v_0),k\}}$, either way it maps (by $\chi \circ \psi$) to a closed edge. If $\phi_1(e_i)$ correspond to an element in $Lk(v_0)$ then it just maps to a closed edge e_i in $\mathcal{S}_{\Gamma\{St(v_0),k\}}$. If however, $\phi_1(e_i)$ corresponds to the generator v_0, then is $\chi \circ \psi(e_i) = \bullet \in \mathcal{S}_{\Gamma\{St(v),k\}}$ not a closed edge, however $\chi \circ \psi \left(\underbrace{e_i e_i \cdots e_i}_{k \text{ terms}} \right) = e_i'$ will be a closed edge. Hence, after k edges in the path $p = (e_1, e_2, \ldots, e_l)$ there is at least 1 edge in $\chi \circ \psi(e_1, e_2, \ldots, e_l)$ that is closed. Hence, we have that the amount of edges in $\chi \circ \psi(p)$ is at least $\lfloor \frac{l}{k} \rfloor \geq \frac{l}{k} - (k-1)$. If we now lift $\chi\left(\psi(p)\right)$ to a path starting at $\bullet_0' = \overline{\chi}(\bullet_0)$ in $\overline{\mathcal{S}}_{\Gamma\{St(v_0),k\}}$ this path will end at $\overline{\chi}(g)$, this is by definition of the construction of $\overline{\chi}$ in Lemma 6.9.3, and the lift consist of at least $\frac{l}{k} - (k-1)$ edges. We should be careful even though we proved that a path p of length l in $\overline{\mathcal{S}_\Gamma}$ maps to a path p' of length at least $\frac{l}{k} - (k-1)$ in $\overline{\mathcal{S}}_{\Gamma\{St(v_0),k\}}$, it does not mean p' is a path of minimal length between $\bullet_0'$ and $\overline{\chi}(g)$ in $\overline{\mathcal{S}}_{\Gamma\{St(v_0),k\}}$. Anyway since $\chi \circ \psi$ is surjective every possible path will be reached. We conclude

$$\frac{1}{k} d_{A_\Gamma}(x, \bullet_0) - k \leq d_{A_{\Gamma\{St(v),k\}}}(\overline{\chi}(x), \bullet_0') \leq d_{A_\Gamma}(x, \bullet_0).$$

Hence, we proved that $\overline{\chi}$ satisfies equation (6.1).

To prove that it also satisfies (6.2), it is sufficient to prove that $\overline{\chi}$ is subjective on the set of vertices $\overline{\mathcal{S}}_\Gamma^{(0)} \to \overline{\mathcal{S}}_{\Gamma\{St(v),k\}}^{(0)}$. This is the case since the vertices $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}^{(0)}$ are in bijection with the closed paths in $\mathcal{S}_{\Gamma\{St(v),k\}}^{(0)}$, because χ is a deformation retract it is in bijection with the closed paths in $\overset{k}{\mathcal{S}}_\Gamma$, However since $\overline{\mathcal{S}_\Gamma}$ is also the universal cover of $\overset{k}{\mathcal{S}}_\Gamma$, these closed paths are all reached by paths starting at $\bullet_0$ and ending in a vertex of $\overline{\mathcal{S}_\Gamma}^{(0)}$. One can easily check that the map $\overline{\chi}$ does exactly this surjective (See construction of $\overline{\chi}$ in Lemma 6.9.3).

(iii) part 1: We first prove that the image of a geodesic $l \subseteq \overline{\mathcal{S}_\Gamma}$ under $\overline{\chi}$ is again a geodesic. Let l be a geodesic in $\overline{\mathcal{S}_\Gamma}$ i.e. l is the connected component of the inverse image of a loop $e_i \in \pi_1(\mathcal{S}_g)$ under ϕ_1, such that $l^{(0)} = x^{A_{\{e_i\}}}$ for any $x \in l^{(0)}$ (see Lemma 5.3.12).

- Case 1: $e_i = v_0$, then l will cover the stretched circle in $\overset{k}{\mathcal{S}}_\Gamma$ and hence, $\chi\big(\psi(l)\big)$ will cover a circle (that correspond to e_i but now in $\mathcal{S}_{\Gamma\{St(v),k\}}$, more precisely the lift of k concatenated e_i in $\mathcal{S}_\Gamma$ to $\overline{\mathcal{S}_\Gamma}$ will map by $\psi \circ \chi$ to a circle e_i in $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$).

- Case 2: $e_i \neq v_0$ but $e_i \sim v_0$, then l will cover a circle in $\overset{k}{\mathcal{S}}_\Gamma$ and hence, $\chi\big(\psi(l)\big)$ will cover a circle as well (that correspond to e_i but now in $\mathcal{S}_{\Gamma\{St(v),k\}}$).

- Case 3: $e_i \neq v_0$ and $e_i \not\sim v_0$, then l will cover a circle in $\overset{k}{\mathcal{S}}_\Gamma$ and hence, $\chi\big(\psi(l)\big)$ will cover a circle as well (that correspond to one of the copies $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ in $\mathcal{S}_{\Gamma\{St(v),k\}}$).

Either way we can say that l covers a circle e in $\mathcal{S}_\Gamma$ and $\underline{\chi}(\psi(l))$ is a circle $\tilde{e}$ in $\mathcal{S}_{\Gamma\{St(v_0),k\}}$. Pick $y \in l^{(0)}$ arbitrary, there is a path p in $\overline{\mathcal{S}_\Gamma}$ between x and y that is completely contained in l. Now consider the path $\chi\big(\psi(p)\big)$ in $\mathcal{S}_{\Gamma\{St(v),k\}}$ (this path only consists of concatenated $\tilde{e}$'s). We can lift this path to $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$ starting at $\overline{\chi}(x)$ and is hence contained in a flat l' in $\phi_2^{-1}(\tilde{e})$, by Lemma 6.9.3 the endpoint is $\overline{\chi}(y)$. The flat l' in $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$ that is the connected component in $\phi_2^{-1}(\tilde{e})$ containing $\overline{\chi}(x)$ thus also contains $\overline{\chi}(y)$. We now have that $\overline{\chi}(l) \subseteq l'$. We prove the other inclusion, consider $z \in l'$ let p' be the path in l' from $\overline{\chi}(x)$ to z this path maps by $\underline{\phi_2}$ to say $m \in \mathbb{Z}$ concatenated $\tilde{e}$ in $\mathcal{S}_{\Gamma\{St(v_0),k\}}$. If we then lift $m \cdot e \in \pi_1(\mathcal{S}_\Gamma)$ to $\overline{\mathcal{S}_\Gamma}$ starting at x (in Case 2 & 3), lift $km \cdot e \in \pi_1(\mathcal{S}_\Gamma)$ to $\overline{\mathcal{S}_\Gamma}$ starting at x (in Case 1). Then one can check that the endpoint of this lift is mapped by $\overline{\chi}$ to z.

(iii) part 3: Secondly, We prove that $\overline{\chi}$ is surjective on the set of geodesics. Pick l a geodesic of $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$, that covers $\tilde{e}$. We have proven in **(ii)** that $\overline{\chi}$ is surjective on the set of vertices. Pick $y \in l$ then there is a vertex $x \in \overline{\mathcal{S}_\Gamma}$ such that $\overline{\chi}(x) = y$, one can easily verify that geodesic $x^{A_e} =: l'$ is mapped by $\overline{\chi}$ to

l. Hence, $\overline{\chi}$ induces a subjection between geodesics. Completely similar we can prove that $\overline{\chi}$ sends 2 dimensional flats to 2 dimensional flats, and prove that $\overline{\chi}$ is surjective on the set of 2-flats.

(iii) part 3: We prove that $\overline{\chi}$ maps parallel classes to parallel classes. Two geodesics l and l' are parallel if there is a sequence of geodesics $l = l_2, l_2, \dots, l_m = l'$ such that l_i and l_{i+1} are parallel in the same 2 dimensional flat (i.e. in $\mathbb{Z} \times \mathbb{Z}$). Hence, we only have to prove that if l and l' are parallel in a 2 dimensional flat, then $\overline{\chi}(l)$ and $\overline{\chi}(l')$ are parallel in a 2 dimensional flat.

So suppose l and l' are in the same 2-flat F, hence that $\phi_1(l) = \phi_1(l) = e_1 \in \pi_1(\mathcal{S}_\Gamma)$ for a certain loop e_1. This loop corresponds with a generator in $V(\Gamma)$. Suppose $\phi_1(F) = e_1 \times e_2$ a certain 2 dimensional torus.

We saw that $\overline{\chi}$ sends 2 dimensional flats to 2 dimensional flats, and that $\overline{\chi}$ is surjective on the set of 2-flats. Hence, $\phi_2\left(\overline{\chi}(F)\right) = \tilde{e}_1 \times \tilde{e}_2$ for certain two generators in $\pi_1(\mathcal{S}_{\Gamma\{St(v),k\}})$. Hence, we just have to prove that $\overline{\chi}(l)$ and $\overline{\chi}(l')$ are either disjoint or coincide (which is equivalent of being parallel in $\mathbb{R}^2$). Suppose not then $\overline{\chi}(l)$ covers $\tilde{e}_1$ in $\mathcal{S}_{\Gamma\{St(v),k\}}$ and $\overline{\chi}(l')$ covers $\tilde{e}_2$ in $\mathcal{S}_{\Gamma\{St(v),k\}}$. However, we know that both $\overline{\chi}(l)$ and $\overline{\chi}(l')$ covers an edge that corresponds to e_1 or to one of its k copies $e_{1_1}, e_{1_2}, \dots, e_{1_k}$ in $\mathcal{S}_{\Gamma St(v_0),k)}$ (if $e_1 \neq v_0$ and $e_1 \not\sim v_0$) . Either way the generators of these copies are never adjacent in $\Gamma\{St(v_0), k\}$, while $\tilde{e}_1 \sim \tilde{e}_2$ are adjacent in $\Gamma\{St(v_0), k\}$. We conclude that $\phi_2\left(\overline{\chi}(l)\right) = \tilde{e}_1 = \phi_2\left(\overline{\chi}(l)\right)$.

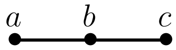
Denote $[l]$ as the parallel class of a geodesic, then we have proven that $\overline{\chi}\left([l]\right) \subseteq [\overline{\chi}(l)]$. Suppose $\overline{\chi}\left([l]\right) \subsetneq [\overline{\chi}(l)]$. Then there is at least one pair $\overline{\chi}(l_1) \in \overline{\chi}\left([l]\right)$ and $\overline{\chi}(l_2) \in [\overline{\chi}(l)] \setminus \overline{\chi}\left([l]\right)$ such that they are parallel in a 2-flat being $\overline{\chi}(F)$, hence we can choose $l_1', l_2' \in F$ satisfying the same (since $\overline{\chi}$ is surjective on the set of geodesics). However now it follows form previous discussion that if l_1' and l_2' are not parallel then $\overline{\chi}(l_1')$ and $\overline{\chi}(l_2')$ are not. Hence, we have proven that $\overline{\chi}$ induces a bijection between parallel classes.

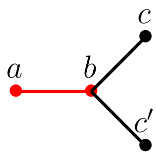
(iii) part 4: We prove that if l and l' spans a 2 flat in $\overline{\mathcal{S}_\Gamma}$ (i.e. they are contained in a 2-flat but not parallel) then $\overline{\chi}(l)$ and $\overline{\chi}(l')$ span a 2 flat in $\overline{\mathcal{S}}_{\Gamma\{St(v),k\}}$. This follows from part 2, and since there we saw that 2 flats mapped to 2 flats.
Until now, we have proven that $\overline{\chi}$ induces an embedding from $\mathcal{P}(\Gamma)$ to $\mathcal{P}(\Gamma\{St(v), k\})$.

(iii) part 5: Finely we prove that $\overline{\chi}$ induces an isomorphism between the extension complexes. We know that 2-flats map to 2-flats, hence if $[l]$ and $[l']$ are connected in $\mathcal{P}(\Gamma)$ then $[\overline{\chi}(l)]$ and $[\overline{\chi}(l')]$ are connected in $\mathcal{P}(\Gamma\{St(v_0), k\})$. Suppose there is a pair of parallel classes $[l]$ and $[l']$ such that $[l]$ and $[l']$ is not connected in $\mathcal{P}(\Gamma)$ but $\overline{\chi}\left([l]\right) = [\overline{\chi}(l)]$ and $\overline{\chi}\left([l']\right) = [\overline{\chi}(l')]$ are connected in $\mathcal{P}(\Gamma\{St(v_0), k\})$, then there are $\overline{\chi}(l_1) \in \overline{\chi}\left([l]\right)$ and $\overline{\chi}(l_2) \in \overline{\chi}\left([l']\right)$ that span a certain 2-flat call this $\overline{\chi}(F)$. We now look at the 2-flat F in $\overline{\mathcal{S}_\Gamma}$, take any two geodesics in this two flat l_1' and l_2' that span this flat (i.e. $l_1' \cap l_2'$ is a singleton). Then $\overline{\chi}(l_1')$ and $\overline{\chi}(l_2')$ span the 2-flat $\overline{\chi}(F)$ in $\overline{\mathcal{S}}_{\Gamma\{St(v_0),k\}}$. Without loss of generality $\overline{\chi}(l_1')$ is parallel with $\overline{\chi}(l_1)$ and $\overline{\chi}(l_2')$ with $\overline{\chi}(l_2)$. Hence $\overline{\chi}(l_1) \in [\overline{\chi}(l)]$ and $\overline{\chi}(l_2) \in [\overline{\chi}(l')] = \overline{\chi}\left([l]\right)$, we conclude $l_1 \in [l]$ and $l_2 \in [l']$ such that $[l]$ and $[l']$ are connected in $\mathcal{P}(\Gamma)$. $\square$

Remark 6.9.5. Notice that in Theorem 6.9.4 we not only constructed quasi-isometries between some right-angled Artin groups, we also proved that their extension complexes are isomorphic. Hence, we again have a partial answer to the question described in Remark 6.3.6.

To get some intuition in how these constructions work we give some examples.

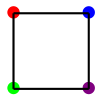
Example 6.9.6. (i) Consider $P_2 :=$ 

, we will double around the star of the first vertex, hence we get $P_2\{st(\cdot), 2\} =$ 

. The 2-fold cover $\overset{2}{\mathcal{S}}_{P_3} = \mathcal{S}_{p_3}/\ker(\phi)$ of $\mathcal{S}_\Gamma$ where

$$\begin{aligned} \phi : A_{P_3} &\to \mathbb{Z}/2\mathbb{Z} : \\ a &\mapsto 1, \\ b, c &\mapsto 0, \end{aligned}$$

is precisely the first complex in Figure 6.9.4.

(ii) We apply the construction in Theorem 6.9.4 to $\Gamma :=$  around the vertex $v :=$●. In Figure 6.9.5, the Salvetti complex of Γ is drawn as well as the Salvetti complex of $\Gamma_0 := \Gamma \setminus St(●)$, the space L (see proof Theorem 6.9.4), the k-fold cover and the Salvetti complex of $\Gamma\{St(●), 2\}$ drawn.

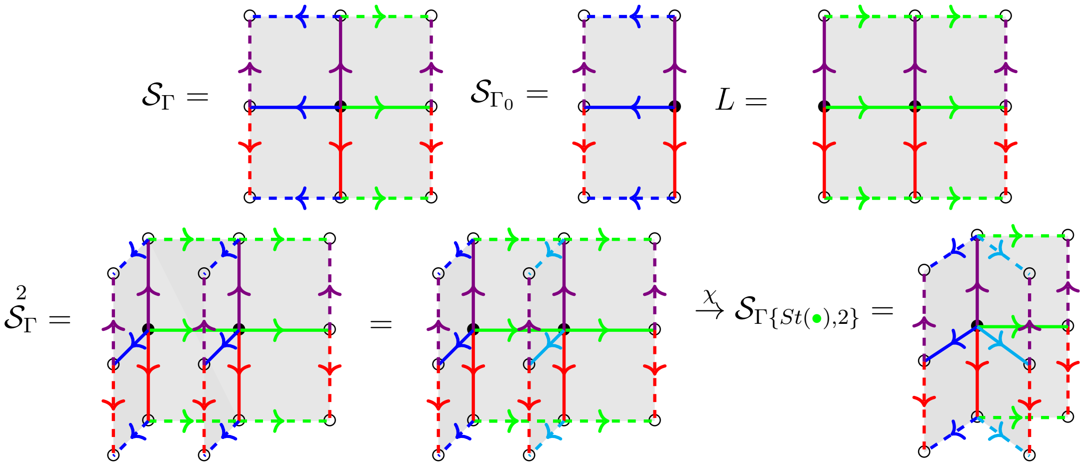
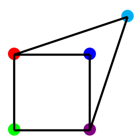


Figure 6.9.5: Double procedure for Γ.

We get $\Gamma\{st(●, 2)\} :=$  .

(iii) At last, we will show for an easy example what $\overline{\chi}$ does to the universal cover of $\mathcal{S}_\Gamma$, where $\Gamma :=$ • • . If we double this graph around a vertex we get $\Gamma\{St(\cdot), 2\} :=$ • • • , hence we have that the free group of rank 3 is an index 2 subgroup of the free group of rank 2. The universal cover is drawn in Figure 6.9.6. We see that every other edge (corresponding to one generator) is being contracted in an alternating way to one vertex.

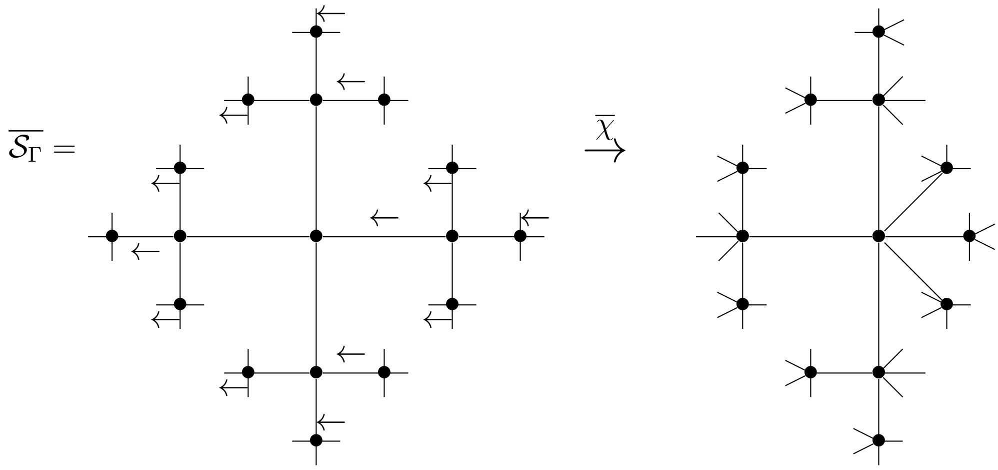


Figure 6.9.6: From the Cayleygraph of rank 2 free group to the Cayleygraph of the rank 3 free group.

Remark 6.9.7. (i) In Theorem 6.9.4 we found for every right-angled Artin group A_Γ with diameter of the defining graph at least 3, a quasi-isomorphic right-angled Artin group that is not isomorphic to A_Γ (Theorem 6.1.3). Since if a graph Γ has diameter at least 3, and we double around a vertex v_0 such that $\max\{d(v_0, w) \mid w \in V(\Gamma)\} \geq 3$, then the graph $\Gamma\{St(v_0), k\}$ will not be isomorphic to Γ. This however does not conflict with theorems like Theorem 6.5.2 and Theorem 6.6.3, where we proved that some classes of Artin groups are quasi-isometrically rigid, but since applying a k-double around a star we always create a graph that does not satisfy Definition 6.5.1(ii) or Property 6.4.1 (iv) we never obtain an Artin group in one of these classes.

(ii) From Theorem 6.9.4 (ii) it follows that the group A_Γ is quasi-isometric to $A_{\Gamma\{St(v),k\}}$. However, this in itself already follows from the fact that $A_{\Gamma\{St(v),k\}}$ is a finite index (that being k) subgroup of A_Γ and hence, quasi-isomorphic.

(iii) Theorem 6.9.4 gives a lot of quasi-isometric non-isomorphic RAAGs. However, even though for every tree T of diameter at least 3, we can construct a tree T' by applying this Theorem multiple times to $P_4 :=$ •—•—•—• such that T' will contain T as a subtree, this is not sufficient to prove Theorem 6.8.1. It only proves that there is a quasi-isometric embedding between

A_T and A_{P_4} by Lemma 6.1.4. There is of course also a quasi-isometric embedding from A_{P_4} to A_T by the same lemma. However, embeddings in both directions don't always imply that the spaces are quasi-isometric.

One could suspect that maybe by using only Theorem 6.9.4 one can prove Theorem 6.8.1. Suppose you have a tree Γ of diameter at least 3, and suppose by apply finite amount of times Theorem 6.9.4 one gets a tree Γ' such that this tree can also be constructed by applying finite amount of times Theorem 6.9.4 on P_4. Then we conclude that A_Γ and A_{P_4} are quasi-isometric, since $A_\Gamma \underset{q.i.}{\cong} A_{\Gamma'} \underset{q.i.}{\cong} A_{P_4}$. However, the following Lemma proves this is not always possible.

Theorem 6.9.8. *Let Γ a graph and $v \in V(\Gamma)$ a vertex, denote $l(v)$ to be the minimum distance between v and a leaf*[28] *of Γ. After applying Theorem 6.9.4 on Γ, $l(v)$ will be unchanged.*

Proof. First of all observe yourself that we can assume that Γ is connected. Take $v, w \in V(\Gamma)$ (we allow $v = w$), denote $\tilde{\Gamma}$ to be the graph obtained by a k-doubling around the star $St(w)$. Denote

$$l_\Gamma(v) := \min\{d(v, g) \mid g \text{ is a leaf of } \Gamma\};$$
$$l_{\tilde{\Gamma}}(v) := \min\{d(v, g) \mid g \text{ is a leaf of } \tilde{\Gamma}\}.$$

Since Γ is a subgraph of $\tilde{\Gamma}$ we have $l_{\tilde{\Gamma}}(v) \leq l_\Gamma(v)$. Suppose $l_{\tilde{\Gamma}}(v) \lneq l_\Gamma(v)$, and suppose $\tilde{v} \in V(\tilde{\Gamma})$ is a leaf such that $d(v, \tilde{v}) = l_{\tilde{\Gamma}}(v)$. Suppose $\Gamma \setminus St(w)$ has n components, then $\tilde{\Gamma} \setminus St(w)$ has $k \cdot n$-components. Denote C_1 for the set of n components of $\tilde{\Gamma} \setminus St(w)$ which already existed in $\Gamma \setminus St(w)$. Denote C_2 for the set of $kn - n$ new components of $\tilde{\Gamma} \setminus St(w)$ that did not exist in $\Gamma \setminus St(w)$. Let p be the shortest path from v to $\tilde{v}$ in $\tilde{\Gamma}$. This path starts in a component c of C_1. If we never leave this component then $p \subseteq \Gamma$, and we are done. Hence, at some point we go through the star $St(w)$ and go to another component $\tilde{c}$. We claim that these are the only two components in which p is contained. Suppose not, then we encounter at least 3 components, if so we go at least 2-times through the star $St(w)$. However, then we can skip one of the components (and make our path shorter) in between the first and last component, this new path will be shorter since a minimal path between two vertices in the star $St(w)$ can always be contained in the star itself (See Definition 2.2.5). Hence, we encounter $St(w)$ only once in p, to go from component c to $\tilde{c}$, this also implies that $\tilde{v} \in \tilde{c}$. This component $\tilde{c}$ is thus a component of C_2. If we now look at the component $c' \in C_1$ from which $\tilde{c}$ is copy. This component has the same connection vertices on $St(v)$ and hence we can look at this component and pick the corresponding v' (for which $\tilde{v}$ is the copy in $\tilde{c}$ and hence, is also a leaf), and change p by the path where we replace $p \cap \tilde{c}$ by its copy in c'. We have found a path from v to v' of the same length as p, hence, $l_{\tilde{\Gamma}}(v) \geq l_\Gamma(v)$. □

[28]i.e. $l(v) := \min\{d(v, g) \mid g \text{ is a leaf of } \Gamma\}$

Corollary 6.9.9. *Let L_Γ be the set of all minimal distances to leafs of vertices in Γ (i.e. $L_\Gamma := \{l_\Gamma(v) \mid v \in \Gamma\}$). Suppose a graph $\tilde{\Gamma}$ is constructed from Γ by applying Theorem 6.9.4. Then $L_\Gamma = L_{\tilde{\Gamma}}$.*

Proof. Take $\tilde{v} \in V(\tilde{\Gamma})$, if[29] $\tilde{v} \in V(\Gamma)$ then we are done by Theorem 6.9.8. If $\tilde{v} \in V(\tilde{\Gamma}) \setminus V(\Gamma)$, then there exists $v \in V(\Gamma)$ such that $\tilde{v}$ is one of its (k) doubles. It is easy to see that $L_{\tilde{g}}(\tilde{v}) = L_{\tilde{g}}(v)(= l_\Gamma(v))$, since if we change of the idea of what the subgraph that was the original graph is from Γ that contains v to Γ' that contains $\tilde{v}$ (and $\Gamma' \cong \Gamma$) we get what we wanted. □

Corollary 6.9.10. *There exists no graph, such that it can be constructed from both P_4 and P_5 by applying Theorem 6.9.4.*

Proof. The graph $P_5 :=$ contains a vertex such that the minimum distance to a leaf is $2 = l(v)$, while this does not exist in $P_4 :=$. □

It would be very nice to get some generalization of Theorem 6.9.4 to non-right-angled Artin groups. Maybe if we only ask that if $St(v)$ is a right-angled subgraph a proof is mutatis mutandis.

In Corollary 6.9.10 we see that we cannot find Γ_3 such that we can construct this form both P_4 and P_5 even though they are quasi-isometric, however the extension complexes where not even isomorphic (see Remark 6.3.6). So we actually did not need 6.9.8, since Theorem 6.9.4 (iii) already told us that the extension complexes need to be the same. Nevertheless, it is still interesting to know this property is preserved.

[29] Here we see Γ again as a subgraph of $\tilde{\Gamma}$

Conclusion and discussion

In this thesis we saw complexes associated to Artin groups. We proved that the Deligne complex is a building in the right-angled case. We saw that these complexes occurs again in the $K(\pi, 1)$ conjecture and in the study of quasi-isometric right-angled Artin groups. Many result of quasi-isometric right-angled Artin groups are not touched in this thesis. Neither are here any result about quasi-isometric non-right-angled Artin groups discussed. It could be interesting in further research in what extend Theorem 6.9.4 is true for non-right-angled Artin groups. For right-angled Artin groups one could try to find the class of groups for which it is satisfied that they are quasi-isometric if and only if their extension complexes are isomorphic. We proved that the Deligne complex for non-right-angled Artin groups is not a building. However, since it looks like a building (in the sense that it is made out of apartments of the same type), it would be interesting in what extend the results in building theory are true for these complexes.

Nederlandse samenvatting

Geometrie en Topologie van Artin Groepen (Geometry and topology of Artin groups)

In deze masterproef worden Artin groepen onderzocht, met een nadruk op rechthoekige Artin groepen. Hierbij staat de wisselwerking tussen de algebraïsche structuren en hun geometrische realisaties centraal. Deze start met een introductie van Coxeter groepen en hun Tits-representaties, die de basis vormt voor de constructie van het *Davis complex*. Gebruikmakend van dit complex construeert men het *Salvetti complex*, een topologische ruimte waarvan de fundamentele groep juist de Artin groep is die geassocieerd is aan de Coxeter groep waarvan men vertrekt. Een analoog concept van het Davis complex voor een Coxeter groep is het *Deligne complex* voor een Artin groep. Dit complex is voor een rechthoekige Artin groep een equivalent object als een (Tits-)gebouw, waarvan de kamers de elementen in de Artin groep zijn. Ook bespreken we het onopgelost probleem de $K(\pi, 1)$-conjecture die een link heeft met beide het Salvetti complex en het Deligne complex. Verder bekijken we voor rechthoekige Artin groepen structuren zoals het *exploded Salvetti complex* en het *extension complex*. De universal cover van het exploded Salvetti complex is hier, net als dat van het Salvetti complex een CAT(0)-cube complex en het heeft een natuurlijk verband met de geometrische realisatie van het geassocieerde gebouw. Het extension complex is interessant in de studie van quasi-isometrische classificatie van rechthoekige Artin groepen.

Bibliography

[1] Peter Abramenko and Kenneth S. Brown. *Buildings. Theory and applications.*, volume 248 of *Grad. Texts Math.* Berlin: Springer, 2008.

[2] Mahima Ranjan Adhikari. *Basic algebraic topology and its applications.* New Delhi: Springer, 2016.

[3] Joseph A. Altobelli. The word problem for Artin groups of FC type. *J. Pure Appl. Algebra*, 129(1):1–22, 1998.

[4] Yago Antolín and Islam Foniqi. Subgroups of even Artin groups of FC type. *J. Group Theory*, 28(2):431–462, 2025.

[5] Jason A. Behrstock, Tadeusz Januszkiewicz, and Walter D. Neumann. Quasi-isometric classification of some high dimensional right-angled Artin groups. *Groups Geom. Dyn.*, 4(4):681–692, 2010.

[6] Jason A. Behrstock and Walter D. Neumann. Quasi-isometric classification of graph manifold groups. *Duke Math. J.*, 141(2):217–240, 2008.

[7] Robert W. Bell. Combinatorial methods for detecting surface subgroups in right-angled Artin groups. *ISRN Algebra*, 2011:6, 2011. Id/No 102029.

[8] Oussama Bensaid. personal communication.

[9] Mladen Bestvina, Bruce Kleiner, and Michah Sageev. The asymptotic geometry of right-angled Artin groups, I. Preprint, arXiv:0708.1937 [math.GR] (2007), 2007.

[10] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Grad. Texts Math.* New York, NY: Springer, 2005.

[11] Rubén Blasco-García, María Cumplido, and Rose Morris-Wright. The Word Problem is Solvable for 3-free Artin groups in Quadratic Time. Preprint, arXiv:2204.03523 [math.GR] (2022), 2022.

[12] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren Math. Wiss.* Berlin: Springer, 1999.

[13] Egbert Brieskorn. Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Invent. Math.*, 12:57–61, 1971.

[14] Pierre-Emmanuel Caprace. Lectures on proper CAT(0) spaces and their isometry groups. In *Geometric group theory. Lecture notes from the IAS/Park City Mathematics Institute (PCMI) Graduate Summer School, Princeton, NJ, USA, 2012.*, pages 91–125. Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Study (IAS), 2014.

[15] Pierre-Emmanuel Caprace and Tom De Medts. Lattice envelopes of right-angled Artin groups. Preprint, arXiv:2401.15943 [math.GR] (2024), 2024.

[16] Pierre-Emmanuel Caprace and Michah Sageev. Rank rigidity for CAT(0) cube complexes. *Geom. Funct. Anal.*, 21(4):851–891, 2011.

[17] Ruth Charney. The Deligne complex for the four-strand braid group. *Trans. Am. Math. Soc.*, 356(10):3881–3897, 2004.

[18] Ruth Charney. An introduction to right-angled Artin groups. *Geom. Dedicata*, 125:141–158, 2007.

[19] Ruth Charney. Ruth charney - 1/2 artin groups. Institut des Hautes Etudes Scientifiques (IHES), `https://www.youtube.com/watch?v=8B7lgAKvC8g&t=2307s`, 2019.

[20] Ruth Charney and John Crisp. Relative hyperbolicity and Artin groups. *Geom. Dedicata*, 129:1–13, 2007.

[21] Ruth Charney and Michael W. Davis. The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups. *J. Am. Math. Soc.*, 8(3):597–627, 1995.

[22] H. S. M. Coxeter. The complete enumeration of finite groups of the form ${R_i}^2 = (R_iR_j)^{k_{ij}} = 1$. *J. Lond. Math. Soc.*, 10:21–25, 1935.

[23] Michael W. Davis. *Buildings are CAT*(0), pages 108–123. Cambridge: Cambridge University Press, 1998.

[24] Michael W. Davis. *The geometry and topology of Coxeter groups.*, volume 32 of *Lond. Math. Soc. Monogr. Ser.* Princeton, NJ: Princeton University Press, 2008.

[25] Pierre Deligne. Les immeubles des groupes de tresses généralises. *Invent. Math.*, 17:273–302, 1972.

[26] Carl Droms. Isomorphisms of graph groups. *Proc. Am. Math. Soc.*, 100:407–408, 1987.

[27] Cornelia Druţu and Michael Kapovich. *Geometric group theory. With an appendix by Bogdan Nica*, volume 63 of *Colloq. Publ., Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2018.

[28] Eddy Godelle. Artin-Tits groups with CAT(0) Deligne complex. *J. Pure Appl. Algebra*, 208(1):39–52, 2007.

[29] Brayton Gray. *Homotopy theory. An introduction to algebraic topology*, volume 64 of *Pure Appl. Math., Academic Press.* Academic Press, New York, NY, 1975.

[30] Frédéric Haglund and Frédéric Paulin. Arborescent constructions of buildings. *Math. Ann.*, 325(1):137–164, 2003.

[31] Jingyin Huang. Quasi-isometry classification of right-angled Artin groups II: several infinite out cases. Preprint, arXiv:1603.02372 [math.GT] (2016), 2016.

[32] Jingyin Huang. Quasi-isometric classification of right-angled Artin groups. I: The finite out case. *Geom. Topol.*, 21(6):3467–3537, 2017.

[33] Jingyin Huang and Bruce Kleiner. Groups quasi-isometric to right-angled Artin groups. *Duke Math. J.*, 167(3):537–602, 2018.

[34] Sang-Hyun Kim and Thomas Koberda. Embeddability between right-angled Artin groups.. *Geom. Topol.*, 17(1):493–530, 2013.

[35] Ian J. Leary. A metric Kan-Thurston theorem. *J. Topol.*, 6(1):251–284, 2013.

[36] Clara Löh. *Geometric group theory. An introduction.* Universitext. Cham: Springer, 2017.

[37] Alexander Margolis. Quasi-isometry classification of right-angled Artin groups that split over cyclic subgroups. *Groups Geom. Dyn.*, 14(4):1351–1417, 2020.

[38] Alexandre Martin. The Tits alternative for two-dimensional Artin groups and Wise's power alternative. *J. Algebra*, 656:294–323, 2024.

[39] Jon McCammond. The mysterious geometry of Artin groups. *Winter Braids Lect. Notes*, 4:ex, 2017.

[40] Rose Morris-Wright. Parabolic subgroups in FC-type Artin groups. *J. Pure Appl. Algebra*, 225(1):12, 2021. Id/No 106468.

[41] James R. Munkres. *Topology.* Upper Saddle River, NJ: Prentice Hall, 2nd ed. edition, 2000.

[42] Luis Paris. Lectures on Artin groups and the $K(\pi, 1)$ conjecture. In *Groups of exceptional type, Coxeter groups and related geometries. Invited articles based on the presentations at the international conference on "Groups and*

geometries", Bangalore, India, December 10–21, 2012., pages 239–257. New Delhi: Springer, 2014.

[43] Martin Roller. Poc Sets, Median Algebras and Group Actions. Preprint, arXiv:1607.07747 [math.GN] (2016), 2016.

[44] Mark Ronan. *Lectures on buildings.* Chicago, IL: University of Chicago Press, 2nd updated and revised ed. edition, 2009.

[45] Michah Sageev. CAT(0) cube complexes and groups. In *Geometric group theory. Lecture notes from the IAS/Park City Mathematics Institute (PCMI) Graduate Summer School, Princeton, NJ, USA, 2012.*, pages 7–54. Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Study (IAS), 2014.

[46] Petra Schwer. *CAT(0) cube complexes. An introduction*, volume 2324 of *Lect. Notes Math.* Cham: Springer, 2023.

[47] Herman Servatius. Automorphisms of graph groups. *J. Algebra*, 126(1):34–60, 1989.

[48] J. Stillwell. *Classical Topology and Combinatorial Goup Theory second Edition.* Springer-Verlag, New York Berlin Heidelberg, (1993).

[49] Harm van der Lek. Extended Artin groups. Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math. 40, Part 2, 117-121 (1983)., 1983.

[50] Harm van der Lek. The homotopy type of complex hyperplane complements. Ph.D. Thesis, University of Nijmegen, 1983.

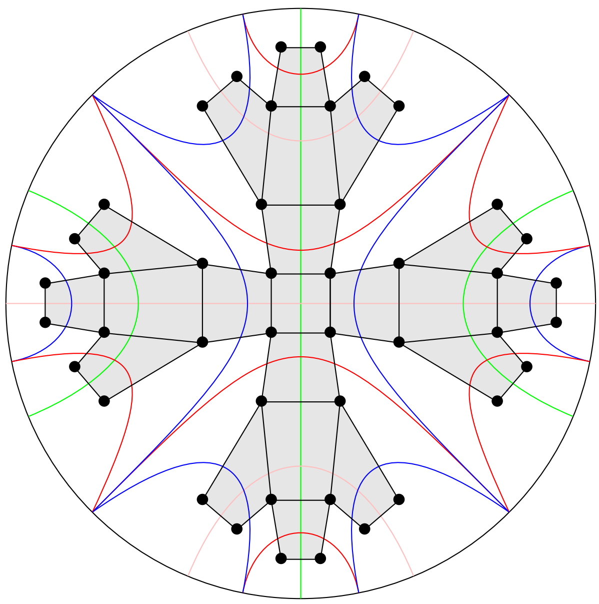


Figure 6.9.7: Davis complex of type $P_4 :=$ •—•—•—•.